CONTINUITY PROBLEM FOR BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS WITH SINGULAR NONMARKOVIAN TERMINAL CONDITIONS AND RANDOM TERMINAL TIMES

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ABSTRACT

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We study a class of nonlinear BSDEs with a superlinear driver process f adapted to a filtration $\mathbb{F}$ and over a random time interval $[0, S]$ where S is a stopping time of $\mathbb{F}$. The filtration is assumed to support at least a d-dimensional Brownian motion as well as a Poisson random measure. The terminal condition ξ is allowed to take the value $+\infty$, i.e., singular. Our goal is to show existence of solutions to the BSDE in this setting. We will do so by proving that the minimal supersolution to the BSDE is a solution, i.e., attains the terminal values with probability 1. We focus on non-Markovian terminal conditions of the following form: 1) $\xi_1 = \infty \cdot \mathbf{1}_{\{\tau \leq S\}}$ and 2) $\xi_2 = \infty \cdot \mathbf{1}_{\{\tau > S\}}$ where τ is another stopping time. We call a stopping time S solvable with respect to a given BSDE and filtration if the BSDE has a minimal supersolution with terminal value ∞ at terminal time S. The concept of solvability plays a key role in many of the arguments. We also use the solvability concept to relax integribility conditions assumed in previous works for continuity results for BSDE with singular terminal conditions for terminal values of the form $\infty \cdot \mathbf{1}_{\{\tau \leq T\}}$ where T is deterministic. We provide numerical examples in cases where the solution is explicitly computable and a basic application in optimal liquidation.

Keywords: BSDE in Finanance, Non Markovian Singular Terminal Value, Control Problem, Continuity Problem

ÖZ

MARKOV OLMAYAN TEKİL SON DEĞERLİ VE RASTGELE SON ZAMANLI GERİYE DOĞRU STOKASTİK DİFERANSİYEL DENKLEMLER İÇİN SÜREKLİLİK PROBLEMİ

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$\mathbb{F} = \{\mathcal{F}_t, t \in [0, \infty)\}$ en azından d-boyutlu bir Brown hareketi ve $\mathbb{R}^m \setminus \{0\}$ üzerinde bir Poisson rastgele-ölçümü kapsayan bir filtrasyon, S bu filtrasyonun bir durma zamanı olsun. Bir Markov sürecin bir kümeden ilk çıktığı andaki pozisyonunun deterministik bir fonksiyonu olan rastgele değişkenlere "Markov," ∞ değerini alabilen rastgele değişkenlere de "tekil" (singular) diyelim. Bu tezin amacı, rastgele $[0, S]$ zaman aralığında, $\mathbb{F}$ filtrasyonuyla uyumlu (adapted) doğrusal-üstü (super-linear) sürücü bir f sürecinin tanımladığı geriye doğru stokastik diferansiyel denklemin (Backward stochastic differential equations (BSDE)) tekil ve Markov olmayan son değerler için çözümlerini çalışmaktır. τ, $\mathbb{F}$ filtrasyonunun başka bir durma zamanı olsun. İki

çeşit son değer için çözüm varlığı ispatlanmıştır: $\xi_1 = \infty \cdot \mathbf{1}_{\{\tau \leq S\}}$ ve $\xi_2 = \infty \cdot \mathbf{1}_{\{\tau > S\}}$. Bu son değerler için çözüm varlığı, aynı son değerler için var olduğu bilinen minimal üstçözümlerin S'de sürekli oldukları ve limitlerinin son değere eşit olduğu ispat edilerek gösterilmiştir. BSDE'nin S anında ∞ değerini alan bir üstçözümü varsa S'ye "çözülebilir" (solvable) dedik. Tezimizdeki argümanların birçoğu bu kavram üzerine kuruludur. Bu kavram kullanılarak geçmişte elde edilen bazı tekil son değerli çözümlerin daha genel şartlar altında bulunabileceği de gösterilmiştir. Çözümlerin açık olarak ifade edilebileceği durumlar için sayısal örnekler ve çözümlerin simülasyon grafikleri verilmiştir. Son olarak çalıştığımız BSDE'lerin matematiksel finansta optimal pozisyon kapatma sorusuna bir uygulaması anlatılmıştır.

Anahtar Kelimeler: Finansta BSDE, Markov olmayan Tekil son değerli, Konrollü Problem, Süreklilik Problemi

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TABLE OF CONTENTS

ABSTRACT vii

ÖZ ix

ACKNOWLEDGMENTS xiii

TABLE OF CONTENTS xv

LIST OF FIGURES xvii

CHAPTERS

1 INTRODUCTION 1

1.1 Outline of results 4

1.2 Implications for PDE, stochastic optimal control and BSDE theory 7

1.3 Integrable data 8

1.4 Supersolutions for singular terminal conditions 13

1.4.1 Additional conditions on f 15

1.5 Known results for exit times 16

2 LITERATURE REVIEW 19

2.1 BSDE and applications in finance 19

2.2 Singular terminal conditions 20

2.3 Continuity results for BSDE with non-Markovian singular terminal conditions 21

2.4 BSDE with singular terminal values and control problems with constraints 23

3 SOLVABLE STOPPING TIME AND MINIMAL SUPERSOLUTION 27

4 TERMINAL CONDITION ξ_1 33

4.1 First approach: The use of a Constructed Auxiliary Upper-bound 33

4.2 A new argument for ξ_1 based on solvability 40

4.3 An example in one space dimension 43

4.4 A numerical example 48

5 TERMINAL CONDITION ξ_2 51

6 AN APPLICATION TO OPTIMAL LIQUIDATION 57

7 CONCLUSION 67

REFERENCES 69

APPENDICES

LIST OF FIGURES

FIGURES

Figure 4.1 Solution for ODE 4.21 when $v_1 = 1$, and $q = 1.4$ 46

Figure 4.2 Simulation 1: The W process remains within the $\pm L$ band. 48

Figure 4.3 Simulation 2: The W process exits the $\pm L$ band and terminates the at time τ. 49

Figure 6.1 Graph of the function $U(s)$ 62

Figure 6.2 Trading Strategy qt with the $S(t)$ process 63

Figure 6.3 Trading Strategy qt does not liquidate completely as the process S remains below u 64

Figure 6.4 Trading Strategy with varying Volatility $(\sigma_1 < \sigma_2 < \sigma_3)$. 65

Figure 6.5 Trading Strategy with Varying Volumes $(V_1 < V_2 < V_3)$. 65

Figure 6.6 Trading Strategy with Varying Opening Position $q0$ 66

Figure 6.7 Trading Strategy with Varying Trading Cost $(\eta_1 < \eta_2 < \eta_3)$. . . . 66

CHAPTER 1

INTRODUCTION

A backward stochastic differential equation (BSDE) is a stochastic differential equation (SDE) with a prescribed terminal condition. They have been intensively studied since the seminal papers [5, 31]; they arise naturally in stochastic optimal control problems (see among others [38]), they provide a probabilistic representation of semilinear partial differential equations (PDE) extending the Feynman-Kac formula ([32]) and they have found numerous applications in finance and insurance [7, 10].

If the driver term of the BSDE has superlinear growth, the solution of the BSDE can blow up in finite time, this allows one to specify ∞ as a possible terminal value for such BSDE; when the terminal value is allowed to take ∞ it is called "singular."

The works [1, 22, 34, 37] study nonlinear BSDE with singular terminal condition at a deterministic terminal time T. Such BSDE generalize parabolic diffusion-reaction PDE with singular boundary conditions and they arise naturally in class of stochastic optimal control problems with terminal constraints [3, 15, 22]; we further comment on this connection in Chapters 2 and 6.

This thesis focuses on BSDE with singular terminal conditions over a random time horizon. We do this within the general framework for BSDE with terminal singular values established in [21, 22, 23] and consider BSDE of the following form

$$dY_t = -f(t, Y_t, Z_t, U_t)dt + Z_t dW_t + \int_{\mathcal{E}} U_t(e)\widetilde{\pi}(de, dt) + dM_t, Y_S = \xi, \qquad (1.1)$$

where W is a d-dimensional Brownian motion and $\widetilde{\pi}$ is a compensated Poisson random measure on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$; the unknown that is sought is the quadruple (Y, Z, U, M). The filtration $\mathbb{F}$ is supposed to be complete and right continuous. The solution component M is required to be a local martingale orthogonal to $\widetilde{\pi}$. The function $f : \Omega \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{B}^2_\mu \to \mathbb{R}$ is called the *generator* (or *driver*) of the BSDE. Finally S is a stopping time of the filtration $\mathbb{F}$ and ξ is an $\mathcal{F}_S$ measurable random variable, which is singular, i.e., $\mathbb{P}(\{\xi = \infty\}) > 0$. Precise conditions on all of these terms are spelled out in sections 1.3 and 1.4 below. A quadruple (Y, Z, U, M) is said to be a supersolution of (1.1) if it satisfies the first equation in (1.1) and

$$\liminf_{t\to+\infty} Y_{t\wedge S} \geq \xi, \text{ almost surely}, \qquad (1.2)$$

holds. A supersolution $(Y^{\min}, Z^{\min}, U^{\min}, M^{\min})$ is called minimal if $Y^{\min} \leq Y$ for any other supersolution (Y, Z, U, M). We say (Y, Z, U, M) solves the BSDE with singular terminal condition ξ if it satisfies the first equation in (1.1) and

$$\lim_{t\to+\infty} Y_{t\wedge S} = \xi; \qquad (1.3)$$

i.e., to go from a supersolution to a solution we need to replace the $\liminf$ in (1.2) with $\lim$ and $\geq$ with $=$. The condition (1.3) means that the process Y is continuous at time S; for this reason we refer to the problem of establishing that a candidate solution satisfies (1.3) as the *"continuity problem"*. This thesis is focused on the study of this problem for the terminal values we describe in the next section. Just as BSDE over deterministic time intervals generalize parabolic PDE, BSDE over random time intervals are generalizations of elliptic PDE; we provide further comments on this connection, on the motivation for the study of BSDE over random time horizon with singular terminal values and on the implication of continuity results for BSDE theory as well as constrained stochastic optimal control at the end of this chapter.

We call a terminal condition “Markovian” if it is of the form $\xi = g(\Xi_S)$ where, $g : \mathbb{R}^d \mapsto \mathbb{R}_+ \cup \{\infty\}$, Ξ is a Markov diffusion process and S is the first time Ξ hits a smooth ∂D, $D \subset \mathbb{R}^d$. For such exit times, existence of minimal supersolutions for (1.1) are proved in [22] for arbitrary terminal condition (see section 1.4 below). The work [35] proves that these minimal supersolutions are in fact solutions for the case $\mathbb{F} = \mathbb{F}^W$ and for the specific generator $f(y) = -y|y|^{q-1}$ and for Markovian terminal conditions. To the best of our knowledge, the continuity problem, that is the existence of the limit $\lim_{t\to+\infty} Y^{\min}_{t\wedge S}$ and the a.s. equality (1.3):

$$\lim_{t\to+\infty} Y^{\min}_{t\wedge S} = \xi \tag{1.4}$$

has been studied only in [35], under the Brownian setting, for $f(y) = -y|y|^{q-1}$ and in the Markovian setting, $\xi = \Phi(\Xi_S)$ and when S is the first exit time of Ξ from a smooth domain. The works [21, 23] develop solutions to (1.1) when ξ belongs to some integrability space. The goal of the present work is to prove that the minimal supersolution of (1.1) satisfies (1.3) (and therefore is a solution) for two classes of *non-Markovian* singular terminal conditions under several assumptions on S. We outline these classes and assumptions in the following paragraphs. Markovian singular terminal conditions are treated in [36, Section 4] within the framework used in this thesis.

1.1 Outline of results

In two previous works [1] and [37] that prove continuity results for deterministic terminal times, two of the main ingredients are the minimal supersolution $Y^{\min,\infty}$ with terminal condition ∞ at terminal time and the apriori upperbounds on supersolutions; both of these, are readily available in the prior literature for deterministic terminal times (for the one dimensional Brownian case treated in [37], $Y^{\min,\infty}$ is deterministic and has an explicit formula). For random terminal times the existence of $Y^{\min,\infty}$ and apriori upperbounds are known only for exit times of Markov diffusions from smooth domains. One of the main ideas of the present work is to impose the existence of $Y^{\min,\infty}$ as an assumption on the stopping time S and base most of our arguments on this assumption. We call the terminal stopping time S *solvable* with respect to the BSDE (1.1) if there exists a supersolution to the BSDE with terminal value ∞ at terminal time S (see Definition 3), deterministic times and exit times of Markov diffusion processes are solvable for a wide range of BSDE; times that have a strictly positive density around 0 are not solvable [22]. Many of our arguments are based on this solvability concept; some basic consequences of solvability are given in Chapter 3. In particular, if S is solvable, the BSDE (1.1) has a minimal supersolution for any singular terminal condition $\xi \geq 0$ (Lemma 1). In addition to S being solvable, in many arguments we assume $\mathbb{F}$ to be left continuous at S as defined in (definition 4) for the following reason. Because the filtration $\mathbb{F}$ is assumed to be general (apriori only completeness and right-continuity is assumed) there is no way to control the jumps of the additional local martingale component M of the solution at the terminal time. To avoid such jumps, we suppose that $\mathbb{F}$ is left-continuous at time S.

We now indicate the main results of the present work.

This thesis focuses on the continuity problem for non-Markovian singular terminal values. Chapters 4 and 5 focus on the continuity problem for non-Markovian terminal conditions of the form $\xi_1 = \infty \cdot \mathbf{1}_{\{\tau \leq S\}}$ (Chapter 4) and $\xi_2 = \infty \cdot \mathbf{1}_{\{\tau > S\}}$ (Chapter 5) where τ is another stopping time of $\mathbb{F}$. The results in these chapters generalize results from [37] (the one dimensional Brownian case) and [1] (the general filtration, driver case) treating same type of terminal conditions where S is assumed to be deterministic. Events of the form $\{\tau \leq S\}$ naturally arise when one modifies constraints on stochastic optimal control problems based on the values the state process of the problem takes. We refer to [1, 37] for more comments on why we pay particular attention to these type of non-Markovian terminal conditions. Solution of the continuity problem for general terminal conditions of the form $\infty \cdot \mathbf{1}_A$ for arbitrary $A \in \mathcal{F}_S$ is an open problem even for the one dimensional Brownian case and S deterministic.

Chapter 4 provides two arguments to prove

$$\lim_{t \to +\infty} Y^{\min}_{t \wedge S} = \xi_1. \tag{1.5}$$

The first one is an adaptation of the argument given for the same type of terminal condition in [1]. It involves the construction of an auxiliary linear process that dominates $Y^{\min}$ and that is known to have the desired limit property at the terminal time S. The main assumption on τ for the construction of the upperbound in [1] is that τ has bounded density at the terminal time; in the current setting this is replaced with the assumption that the random variable $\mathbf{1}_{\{\tau \leq S\}} Y^{\infty}_{\tau}$ has a bounded ϱ-moment for some $\varrho > 1$ (see (4.1)). In Proposition 1 we show that if S is the first exit time of a Markov diffusion from a smooth bounded region and τ is a stopping time independent of S then (4.1) is satisfied. The other main ingredient in the construction of the upperbound process in [1] is the apriori upperbounds on the supersolution of BSDE; in the current context this is replaced by the solvability assumption on S. Section 4.2 presents a new argument for the terminal value ξ_1 that is completely based on the original BSDE (i.e., it doesn't involve the solution of an auxiliary linear BSDE). To simplify arguments this section assumes $\mathbb{F}$ to be generated only by the Brownian motion W. The only assumption on τ is that it be solvable. Let $Y^{\tau,\infty}$ be the supersolution of the BSDE with terminal condition ∞ at terminal time τ. The main idea of this argument is the use of the process $Y^{\tau,\infty}$ as an upperbound to prove (1.5). Working directly with the original BSDE in constructing upperbounds can lead to less stringent conditions on model parameters. As an example, we consider in Section 4.3 the case $S = T$ and $\tau = \inf\{t : |W_t| = L\}$ which was originally studied in [37] using essentially a special case of the argument based on the linear auxiliary process which requires the q parameter in assumption **(B2)** to satisfy $q > 2$. The new proof given Section 4.3 based on the new argument based on solvable stopping times establishes (1.5) for the minimal supersolution assuming only $q > 1$; the proof uses explicit solutions of an associated ODE with singular boundary values. Section 4.4 we present figures of the sample paths of Y based on the explicit solution of the ODE presented in the previous section.

The argument in Chapter 5 that proves that the minimal supersolution corresponding to ξ_2 is in fact a solution follows closely the argument given for the same type of terminal condition in [1] for the case $S = T$ deterministic. The assumptions in this chapter are: S is solvable and $\mathbb{P}(S = \tau) = 0$; no solvability is required for τ. To simplify arguments $\mathbb{F}$ is assumed to be generated by the Brownian motion only.

In Chapter 6 we present a finance application: the question is that of an optimal liquidation (formulated within the classical Almgren-Chriss framework ([16, Chapter

3]), the goal is to achieve full liquidation exactly when the price process hits a target level. The price is assumed to be a simple Brownian and the criterion optimized is the expected cost of closing the position. We also provide several numerical examples of the computed optimal liquidation algorithm.

1.2 Implications for PDE, stochastic optimal control and BSDE theory

BSDE with random terminal times are a generalization of elliptic semi-linear PDE (extension of the Feynman-Kac formula, see [6, 30, 32]). The works [8, 24, 25, 26] show that the solution of some of these PDE can exhibit a singularity of the following form on the boundary of the domain D

$$\lim_{x \to \partial D} u(x) = +\infty. \tag{1.6}$$

This boundary behavior generalizes to

$$\lim_{t \to +\infty} Y_{t \wedge S} = +\infty$$

for BSDE of the form (1.1); the clearest connection between (1.6) and the last condition arises when S is a first hitting time of a Markov diffusion process, this connection is treated in detail in [36, Section 4].

Minimal supersolutions of BSDE of the type (1.1) with ∞-valued terminal values at random terminal times can also be used to express the value function of a class of stochastic optimal control problems over a random time horizon $[[0, S]]$ with terminal constraints of the form $\mathbf{1}_A \cdot q_S = 0$, for some $A \in \mathcal{F}_S$, where q is the controlled process. We discuss this connection further in Chapter 2 and Chapter 6.

Strengthening (1.2) to (1.3) (i.e., going from a supersolution to a solution) has implications both for BSDE theory and for stochastic optimal control applications. Consider two distinct terminal values ξ^1 and ξ^2; with (1.2) it is impossible to tell whether the corresponding minimal supersolutions are distinct. Whereas (1.3) guarantees that distinct solutions Y^1 and Y^2 correspond to distinct terminal values ξ^1 and ξ^2. In stochastic optimal control / finance applications a non-tight optimal control (corresponding to strict inequality in (1.2)) can be interpreted as a strictly super-hedging trading strategy. Continuity results overrule such strategies. For more comments on these points we refer the reader to [1].

The next two sections give the definitions, assumptions and results we employ from previous works (Section 1.3 concerns integrable terminal conditions and Section 1.4 concerns singular terminal values). The only novelty is Definition 3, the definition of a solvable stopping time. Section 1.5 summarizes results that are already available in the current literature on the existence of solutions of BSDE with singular terminal conditions at random terminal times. We comment on possible future work in the Conclusion (Chapter 7).

1.3 Integrable data

Let us start with the definition of solution for BSDE (1.1).

Definition 1 (Classical solution)**.** *A process* $(Y, Z, U, M) = (Y_t, Z_t, U_t, M_t)_{t\geq 0}$*, such that*

- Y *is progressively measurable and càdlàg ,*
- Z *is a predictable process with values in* $\mathbb{R}^d$*,*

- *M is a local martingale orthogonal to W and $\widetilde{\pi}$,*

- *U is also predictable and such that for any $t \geq 0$*

$$\int_0^t \int_{\mathcal{E}} (|U_s(e)|^2 \wedge |U_s(e)|)\mu(de) < +\infty,$$

is a solution to the BSDE (1.1) *with random terminal time S with data $(\xi; f)$ if on the set $\{t \geq S\}$ $Y_t = \xi$ and $Z_t = U_t = M_t = 0$, $\mathbb{P}$-a.s., $t \mapsto f(t, Y_t, Z_t, U_t)\mathbf{1}_{t \leq T}$ belongs to $L^1_{loc}(0, \infty)$ for any $T \geq 0$, the stochastic integrals with respect to W and $\widetilde{\pi}$ are well-defined and, $\mathbb{P}$-a.s., for all $0 \leq t \leq T$,*

$$\begin{aligned} Y_{t\wedge S} &= Y_{T\wedge S} + \int_{t\wedge S}^{T\wedge S} f(u, Y_u, Z_u, \psi_u)du - \int_{t\wedge S}^{T\wedge S} Z_u dW_u \\ &- \int_{t\wedge S}^{T\wedge S} \int_{\mathcal{E}} U_u(e)\widetilde{\pi}(de, du) - \int_{t\wedge S}^{T\wedge S} dM_u. \end{aligned} \tag{1.7}$$

For precise definitions of the stochastic integral with respect to $\widetilde{\pi}$ and orthogonality, we refer to [17].

Theorem 3 of [21, 23], ensures the existence and uniqueness of a classical solution, under some conditions on the terminal value ξ and on the generator f. Let us state these conditions and following them the theorem (Theorem 1 below).

Firstly, the following integrability condition is assumed: for some $r > 1$

$$\mathbb{E}\left[e^{r\rho S}|\xi|^r + \int_0^S e^{r\rho t}|f(t, 0, 0, \mathbf{0})|^r dt\right] < +\infty. \tag{1.8}$$

The constant ρ depends on r and on the generator f (see Remark 2). We suppose that $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^m \times \mathfrak{B}^2_\mu \to \mathbb{R}$ is a random measurable function, such that for any $(y, z, \psi) \in \mathbb{R} \times \mathbb{R}^m \times \mathfrak{B}^2_\mu$, the process $f(t, y, z, \psi)$ is progressively measurable. For notational convenience, we write $f^0_t = f(t, 0, 0, \mathbf{0})$, where $\mathbf{0}$ denotes the function mapping $\mathcal{E}$ to $0 \in \mathbb{R}$. The space $\mathfrak{B}^2_\mu$ is defined[1] as follows:

$$\mathfrak{B}^2_\mu = \begin{cases} \mathbb{L}^2_\mu & \text{if } r \geq 2, \\ \\ \mathbb{L}^1_\mu + \mathbb{L}^2_\mu & \text{if } r < 2, \end{cases}$$

where $\mathbb{L}^p_\mu = \mathbb{L}^p(\mathcal{E}, \mu; \mathbb{R})$ is the set of measurable functions $\psi : \mathcal{E} \to \mathbb{R}$ such that

$$\|\psi\|^p_{\mathbb{L}^p_\mu} = \int_{\mathcal{E}} |\psi(e)|^p \mu(de) < +\infty.$$

The next conditions are adapted from [22]:

(A1) The function $y \mapsto f(t, y, z, \psi)$ is continuous and monotone: there exists $\chi \in \mathbb{R}$ such that a.s. and for any $t \geq 0$ and $z \in \mathbb{R}^m$ and $\psi \in \mathfrak{B}^2_\mu$

$$(f(t, y, z, \psi) - f(t, y', z, \psi))(y - y') \leq \chi(y - y')^2.$$

(A2) For every $j > 0$ and $n \geq 0$, the process

$$\Upsilon_t(j) = \sup_{|y| \leq j} |f(t, y, 0, \mathbf{0}) - f^0_t|$$

is in $L^1((0, n) \times \Omega)$.

(A3) There exists a progressively measurable process $\kappa : \Omega \times [0, T] \times \mathcal{E} \times \mathbb{R} \times \mathbb{R}^m \times (\mathfrak{B}^2_\mu)^2 \to \mathbb{R}$ such that for any (y, z, ψ, ϕ), with $\kappa(\cdot, \cdot, y, z, \phi, \psi) = \kappa^{y,z,\psi,\phi}_\cdot(\cdot)$,

$$f(t, y, z, \psi) - f(t, y, z, \phi) \leq \int_{\mathcal{E}} (\psi(e) - \phi(e)) \kappa^{y,z,\psi,\phi}_t(e) \mu(de),$$

with $\mathbb{P} \otimes Leb \otimes \mu$-a.e. for any (y, z, ψ, ϕ), $-1 \leq \kappa^{y,z,\psi,\phi}_t(e)$ and $|\kappa^{y,\psi,\phi}_t(e)| \leq \vartheta(e)$ where ϑ belongs to the dual space of $\mathfrak{B}^2_\mu$, that is $\mathbb{L}^2_\mu$ or $\mathbb{L}^\infty_\mu \cap \mathbb{L}^2_\mu$.

(A4) There exists a constant L_z such that a.s.

$$|f(t, y, z, \psi) - f(t, y, z', \psi)| \leq L_z |z - z'|$$

[1] For the definition of the sum of two Banach spaces, see for example [20]. The introduction of $\mathfrak{B}^2_\mu$ is motivated in [23].

for any (t, y, z, z', ψ).

Remark 1. *We can replace* **(A3)** *by the Lipschitz condition: there exists a constant* L_ϑ *such that*

$$|f(t,y,z,\psi) - f(t,y,z,\phi)| \leq L_\vartheta \|\psi - \phi\|_{\mathfrak{B}^2_\mu}. \tag{1.9}$$

As explained at the beginning of [21, Section 5], **(A3)** *implies* (1.9) *with* L_ϑ *equal to the norm* $\|\vartheta\|_{(\mathfrak{B}^2_\mu)^*}$ *of* ϑ *in the dual space of* $\mathfrak{B}^2_\mu$*. However* **(A3)** *is sufficient to ensure comparison principle for the solution of BSDEs (see [32, Proposition 5.34], [7, Theorem 3.2.1] or [21, Remark 4]).*

We denote

$$K^2 = \frac{1}{2}(L_z^2 + L_\vartheta^2).$$

Remark 2. *Recall the constants* r *and* ρ *appearing in* (1.8)*. The constant* ρ *satisfies*

$$\rho > \nu = \nu(r) := \begin{cases} \chi + K^2 & \text{if } r \geq 2, \\ \chi + \frac{K^2}{r-1} + \frac{L_\vartheta^2}{\varepsilon(L_\vartheta, r)} & \text{if } r < 2, \end{cases} \tag{1.10}$$

where the constant $0 < \varepsilon(L_\vartheta, r) < r - 1$ *depends only on* L_ϑ *and* r *(see [23], Section 4). The additional term in* ν *disappears if the generator does not depend on the jump part* ψ *(that is, if* $L_\vartheta = 0$*). Even if we can not compute* ε *explicitly, we know that*

$$0 < \varepsilon \leq (r-1)\left(2(\alpha(L_\vartheta, r) + 1)^2 - 1\right)^{-\frac{2-r}{2}},$$

and $\alpha(L_\vartheta, r)$ *has to be chosen such that for any* $x \geq \alpha(L_\vartheta, r)$*,*

$$\frac{1}{2^{r/2}} x^r - 2^{r/2} - 1 - r(2L_\vartheta + 1)x \geq 0.$$

The right side is an increasing function with respect to $r \in (1, 2)$ *and decreasing with respect to* $L_\vartheta \geq 0$*. Hence when* r *is close to one and* L_ϑ *is large,* ε *is very small and thus* ρ *becomes large.*

In [21, 23], a second integrability condition is supposed:

$$\mathbb{E}\left[\int_0^S e^{r\rho t}|f(t,e^{-\nu t}\xi_t,e^{-\nu t}\eta_t,e^{-\nu t}\gamma_t)|^r dt\right] < +\infty, \tag{1.11}$$

where $\xi_t = \mathbb{E}(e^{\nu S}\xi|\mathcal{F}_t)$ and (η,γ,N) are given by the martingale representation:

$$e^{\nu S}\xi = \mathbb{E}(e^{\nu S}\xi) + \int_0^\infty \eta_s dW_s + \int_0^\infty \int_{\mathcal{E}} \gamma_s(e)\widetilde{\pi}(de,ds) + N_S$$

with

$$\mathbb{E}\left[\left(\int_0^\infty |\eta_s|^2 ds + \int_0^\infty \int_{\mathcal{E}} |\gamma_s(e)|^2 \pi(de,ds) + [N]_S\right)^{r/2}\right] < +\infty.$$

We now state [21, 23, Theorem 3]:

Theorem 1. *If* **(A1)** *to* **(A4)** *hold and ξ and f^0 satisfy assumptions* (1.8) *and* (1.11)*, then the BSDE* (1.1) *has a unique solution (Y,Z,U,M) in the sense of Definition 1 such that for any $0 \leq t \leq T$*

$$\mathbb{E}\left[e^{r\rho(t\wedge S)}|Y_{t\wedge S}|^r + \int_0^{T\wedge S} e^{p\rho s}|Y_s|^r ds + \int_0^{T\wedge S} e^{r\rho s}|Y_s|^{r-2}|Z_s|^2 \mathbf{1}_{Y_s\neq 0} ds\right]$$
$$+\mathbb{E}\left[\int_0^{T\wedge S} e^{r\rho s}|Y_{s-}|^{r-2}\mathbf{1}_{Y_{s-}\neq 0} d[M]^c_s\right]$$
$$+\mathbb{E}\left[\int_{t\wedge S}^{T\wedge S}\int_{\mathcal{E}} e^{r\rho s}\left(|Y_{s-}|^2 \vee |Y_{s-}+U_s(e)|^2\right)^{r/2-1}\mathbf{1}_{|Y_{s-}|\vee|Y_{s-}+U_s(e)|\neq 0}|U_s(e)|^2\pi(de,ds)\right]$$
$$+\mathbb{E}\left[\sum_{0<s\leq T\wedge S} e^{r\rho s}|\Delta M_s|^2\left(|Y_{s-}|^2 \vee |Y_{s-}+\Delta M_s|^2\right)^{r/2-1}\mathbf{1}_{|Y_{s-}|\vee|Y_{s-}+\Delta M_s|\neq 0}\right] < +\infty.$$

And

$$\mathbb{E}\left[\left(\int_0^S e^{2\rho s}|Z_s|^2 ds\right)^{r/2} + \left(\int_0^S e^{2\rho s}\int_{\mathcal{E}} |U_s(e)|^2\pi(de,ds)\right)^{r/2} + \left(\int_0^S e^{2\rho s} d[M]_s\right)^{r/2}\right]$$
$$\leq C\mathbb{E}\left[e^{r\rho S}|\xi|^p + \int_0^S e^{r\rho s}|f(s,0,0,\mathbf{0})|^r ds\right].$$

The constant C depends only on r, K and χ.

In general (1.11) is not easy to check. Nonetheless, if ξ is bounded, we can take $\nu = 0$ in (1.11) and assume that:

$$\mathbb{E}\left[\int_0^S e^{r\rho t}|f(t,\xi_t,\eta_t,\gamma_t)|^r dt\right] < +\infty,$$

where $\xi_t = \mathbb{E}(\xi|\mathcal{F}_t)$ and

$$\xi = \mathbb{E}(\xi) + \int_0^\infty \eta_s dW_s + \int_0^\infty \int_{\mathcal{E}} \gamma_s(e)\widetilde{\pi}(de, ds) + N_S.$$

1.4 Supersolutions for singular terminal conditions

To lighten the presentation, *in the rest of the thesis, ξ is supposed to be non-negative.* Theorem 1 gives sufficient conditions to ensure the existence and uniqueness of the classical solution (Y, Z, U, M). When the terminal condition is singular, that is if ξ does not belong to any $\mathbb{L}^p(\Omega)$ for some $p > 1$, we adopt the following definition.

Definition 2 (Supersolution for singular terminal condition)**.** *We say that a quadruple of processes (Y, Z, U, M) is a supersolution to the BSDE* (1.1) *with singular terminal condition $Y_S = \xi \geq 0$ if it satisfies:*

1. *There exists some $\ell > 1$ and an increasing sequence of stopping times S_n converging to S such that for all $n > 0$ and all $t \geq 0$*

$$\mathbb{E}\left[\sup_{r\in[0,t]} |Y_{r\wedge S_n}|^\ell + \left(\int_0^{t\wedge S_n} |Z_r|^2 dr\right)^{\ell/2} \right.$$
$$\left. + \left(\int_0^{t\wedge S_n} \int_{\mathcal{E}} |U_r(e)|^2 \pi(de, dr)\right)^{\ell/2} + [M]^{\ell/2}_{t\wedge S_n}\right] < +\infty;$$

2. *Y is non-negative;*

3. *for all $0 \leq t \leq T$ and $n > 0$:*

$$Y_{t\wedge S_n} = Y_{T\wedge S_n} + \int_{t\wedge S_n}^{T\wedge S_n} f(u, Y_u, Z_u, U_u) du - \int_{t\wedge S_\varepsilon}^{T\wedge S_n} Z_u dW_u$$
$$- \int_{t\wedge S_n}^{T\wedge S_n} \int_{\mathcal{E}} U_u(e)\widetilde{\pi}(de, du) - \int_{t\wedge S_n}^{T\wedge S_n} dM_u. \quad (1.12)$$

4. *On the set $\{t \geq S\}$: $Y_t = \xi, Z = U = M = 0$ a.s. and* (1.2) *holds:*

$$\liminf_{t\to+\infty} Y_{t\wedge S} \geq \xi, \quad a.s.$$

We say that (Y, Z, U, M) *is a minimal supersolution to the BSDE* (1.1) *if for any other supersolution* (Y', Z', U', M') *we have* $Y_t \leq Y'_t$ *a.s. for any* $t > 0$.

Remark 3. *The non-negativity condition can be replaced in general by:* Y *is bounded from below by a process* $\bar{Y}$ *such that* $\mathbb{E} \sup_{t \geq 0} |\bar{Y}_{t \wedge S}|^{\ell} < +\infty$.

The below definition formally introduces the concept of a solvable time; as we already indicated above, we think that it provides a general and natural framework for the study of BSDE (1.1) with singular terminal conditions when the terminal time is random:

Definition 3. *A stopping time* S *will be called solvable with respect to the BSDE* (1.1) *if the filtration* $\mathbb{F}$ *is left-continuous at time* S *and if the BSDE* (1.1) *has a supersolution on the time interval* $[\![0, S]\!]$ *with terminal condition* $Y_S = \infty$ *that is defined as the limit of the solution of the same BSDE with terminal condition equal to the constant* k*, as* k *tends to* ∞.

Most of our arguments will be based on solvable stopping times. From [22], we know that every deterministic time S is solvable provided Conditions **(A)** (given above), and **(B1)**, **(B2)** (given below) hold. Exit times of diffusions from smooth domains provide another example of a solvable stopping time, see Theorem 2 below (a restatement of [22, Theorem 2] in terms of solvable times). [22, Example 1] shows that any stopping time that has a strictly positive density around 0 is non-solvable. Section 3 lists some immediate consequences of the definition above that will be useful in the rest of work.

Definition 4. *Let* $\mathcal{T}$ *denote the set of all stopping times of the filtration* $\mathbb{F}$*. The left limit* $\mathbb{F}_{\sigma-}$ *of a filtration at the stopping time* σ *is defined as*

$$\mathbb{F}_{\sigma-} = \sigma\left(\bigcup_{\tau \in \mathcal{T}, \tau < \sigma} \mathcal{F}_{\tau} \right).$$

We say that $\mathbb{F}$ is left continuous at σ if $\mathbb{F}_{\sigma-} = \mathbb{F}_{\sigma}$.

1.4.1 Additional conditions on f

For a singular terminal value ξ, the conditions (1.8) and (1.11) are false. Hence following [22], we add some hypotheses concerning the generator f and the terminal random time S.

(B1) There exists a constant $q > 1$ and a positive and bounded process η such that for any $y \geq 0$

$$f(t, y, z, \psi) \leq -\frac{y}{\eta_t}|y|^{q-1} + f(t, 0, z, \psi).$$

(B2) The process f^0 is bounded[2].

(B3) There exists $\delta > \delta^*$ such that $\mathbb{E}\left[e^{\delta S}\right] < +\infty$. The constant δ^* depends on χ, L_z and L_ϑ.

(B4) There exists $m > m^*$ such that for any j

$$\mathbb{E}\left[\int_0^S |\Upsilon_t(j)|^m dt\right] < +\infty.$$

The value of m^* depends on χ, L_z and L_ϑ and also on δ and δ^*.

We further suppose that the generator $(t, y) \mapsto -y|y|^{q-1}/\eta_t$ satisfies assumptions **(A)** and **(B)**, which means that η satisfies:

$$\left[\mathbb{E}\int_0^T \frac{1}{\eta_t^m} dt\right] < +\infty. \tag{1.13}$$

[2] ξ is non-negative; in general we should assume that ξ^- is bounded.

The values of δ^* and m^* are given in [22]. Let us simply recall that if $y \mapsto f(t,y,z,\psi)$ is non-increasing, that is for $\chi = 0$, then we have:

$$\delta^* = 2K^2, \quad m^* = \frac{2\delta}{\delta - 2K^2}.$$

1.5 Known results for exit times

Let $(Y^{(k)}, Z^{(k)}, \psi^{(k)}, M^{(k)})$ be the unique solution of the BSDE: for any $t < T$

$$\begin{aligned} Y^{(k)}_{t\wedge S} &= Y^{(k)}_{T\wedge S} + \int_{t\wedge S}^{T\wedge S} f(u, Y^{(k)}_u, Z^{(k)}_u, U^{(k)}_u) du \\ &- \int_{t\wedge S}^{T\wedge S} Z^{(k)}_u dW_u - \int_{t\wedge S}^{T\wedge S} \int_{\mathcal{E}} U^{(k)}_u(e)\widetilde{\pi}(de,du) - \int_{t\wedge S}^{T\wedge S} dM^{(k)}_u, \end{aligned} \tag{1.14}$$

with the truncated terminal condition:

$$\mathbb{P} - \text{a.s., on the set } \{t \geq S\}, \quad Y^{(k)}_t = \xi \wedge k,\ Z^{(k)}_t = U^{(k)}_t = M^{(k)}_t = 0. \tag{1.15}$$

From [22, Proposition 5], under **(A)**, **(B2)**, **(B3)** and **(B4)**, there exists a unique solution $(Y^{(k)}, Z^{(k)}, \psi^{(k)}, M^{(k)})$ to the BSDE (1.14) and (1.15). By the comparison principle for BSDEs, the sequence $Y^{(k)}$ is non-decreasing and converges to some process

$$Y^{\min} \doteq \lim_k Y^{(k)} \tag{1.16}$$

As in the case of deterministic terminal times, the key step in [22] in establishing that $Y^{\min}$ is a minimal supersolution to the BSDE (1.1) is to obtain an a priori estimate on $Y^{(k)}$, independent of the constant k. In terms of the concept of solvable stopping times introduced above in Definition 3, the role of the apriori estimate is to ensure that the stopping time S is solvable. To have such an estimate, [22] restricts attention to the case where S is the first hitting time of a diffusion, namely

$$S = S_D = \inf\{t \geq 0, \quad \Xi_t \notin D\}, \tag{1.17}$$

where the forward process Ξ in $\mathbb{R}^d$ is the solution to the stochastic differential equation

$$d\Xi_t = b(\Xi_t)dt + \sigma(\Xi_t)dW_t \tag{1.18}$$

with some initial value $\Xi_0 \in \mathbb{R}^d$. The functions $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ satisfy a global Lipschitz condition: there exists some $C > 0$ such that

$$\forall x, y \in \mathbb{R}^d \quad \|\sigma(x) - \sigma(y)\| + \|b(x) - b(y)\| \leq C\|x - y\|. \tag{1.19}$$

The domain D is an open bounded subset of $\mathbb{R}^d$, whose boundary is at least of class C^2 (see for example [14, Section 6.2] for the definition of a regular boundary); Ξ_0 is assumed to be fixed and in D.

Note that the condition **(B3)** imposes some implicit hypotheses between the generator f, the set D and the coefficients of the SDE (1.18). The [22, Lemma 2] details some sufficient conditions on the coefficients b and σ.

We introduce the signed distance function dist : $\mathbb{R}^d \to \mathbb{R}$ of D, which is defined by $\text{dist}(x) = \inf_{y\notin D} \|x - y\|$ if $x \in D$ and $\text{dist}(x) = -\inf_{y\in D} \|x - y\|$ if $x \notin D$. [22, Proposition 6] is a Keller-Osserman type inequality (see [18, 29]): there exists a constant C such that:

$$0 \leq Y^{(k)}_{t\wedge S} \leq Y^{\min}_{t\wedge S} \leq \frac{C}{\text{dist}(\Xi_{t\wedge S})^{2(p-1)}}. \tag{1.20}$$

Constant $p > 1$ is the Hölder conjugate of q.

For $n \geq 1$, define

$$S_n = \inf\left\{t \geq 0, \text{dist}(\Xi_t) \leq \frac{1}{n}\right\}, \tag{1.21}$$

where $\text{dist}(\Xi_t)$ denotes the distance between the position of Ξ at time t and the boundary of D. The main result [22, Theorem 2] (expressed in terms of solvable stopping times) is:

Theorem 2. *If S is the exit time given by* (1.17)*, and if $\mathbb{F}$ is left-continuous at time S, under Assumptions* **(A)** *and* **(B)***, S is a solvable stopping time (Definition 3). Moreover there exists a minimal supersolution $(Y^{\min}, Z^{\min}, \psi^{\min}, M^{\min})$ to BSDE* (1.1) *with singular terminal condition $Y_S^{\min} = \xi$ (Definition 2).*

CHAPTER 2

LITERATURE REVIEW

We begin this chapter with a brief discussion of applications of BSDE in mathematical finance.

2.1 BSDE and applications in finance

To the best of our knowledge, linear BSDE appeared first in [5] in the study of stochastic optimal control problems. BSDE as a stand alone concept on its own was introduced in [31] which derived existence and uniqueness results for BSDE of the form

$$dY_t = -f(t, Y_t, Z_t)dt + Z_t dW_t; \qquad Y_T = \xi, \tag{2.1}$$

where f is a Lipschitz function. An early review article from 1997 is [10] which discusses many applications of BSDE to mathematical finance and stochastic optimal control and reviews central results in BSDE theory. A recent book that discusses finance and actuarial applications of BSDE is [7]; we now give several application examples from this work.

In insurance models BSDE can arise as a model for payment processes:

$$P(t) = \int_0^t A(u)du + \int_0^t \int_{\mathbb{R}} C(u,s)N(du,ds) + \xi \mathbf{1}_{t=T}, \tag{2.2}$$

where $A(t)$ models premium payments, $C(t,s)N(dt,ds)$ the claim payments (where N models the times when the claims arise) and the terminal condition of the BSDE ξ corresponds to payments made at maturity.

In finance, the value of a self financing portfolio having Δ_t stocks at time t has the form

$$dY(t) = r(t)\cdot Y(t)dt + \Delta(t)\cdot(\mu(t) - r(t))S(t)dt + \Delta(t)\cdot\sigma(t)S(t)dW(t);$$

we observe that if we choose $f(t, Y_t, Z_t) = -r(t)\cdot Y(t) - \Delta(t)\cdot(\mu(t) - r(t))S(t)$ and $Z(t) = \Delta(t)\cdot\sigma(t)S(t)$ the above process is of the form (2.1). If the self financing portfolio targets a terminal value (as is done in option pricing) we have additionally the terminal condition $Y_T = \xi \in \mathbb{F}_T$, which gives a BSDE.

In all of these works, the terminal conditions ξ is assumed to be integrable; when f is nonlinear and has superlinear growth it turns out that ξ can be allowed to take the value $+\infty$. In this case the terminal value is said to be singular; this type of terminal condition is also the subject of the present thesis. In the next section we discuss the literature on this topic.

2.2 Singular terminal conditions

To the best of our knowledge, the first work to consider BSDE with singular terminal values is [34], which focuses on drivers of the form

$$f(t, Y_t, Z_t) = f(t, Y_t, 0) = -Y_t|Y_t|^q. \tag{2.3}$$

It defines the notion of a minimal supersolution of the BDE (2.1) with the above driver function when the terminal value ξ is singular (i.e, when $\mathbb{P}(\xi = \infty) > 0$) (the notion of a supersolution is defined in the previous chapter). For the proof of continuity at terminal time, [34] focuses on the Markovian case, that is, $\xi = g(\Xi_T)$ where $g : \mathbb{R}^m \to \overline{R^+}$ is a measurable function, $F_\infty = \{g = +\infty\}$ is closed and Ξ is a Markov diffusion process whose drift and volatility satisfy Lipschitz and growth conditions.

Subsequent to [34], [35] extended this analysis to the case when the terminal time is given by hitting times of Ξ.

[21] further extended this analysis to the BSDE of the form:

$$dY_t = -f(t, Y_t, Z_t, \psi_t)dt + Z_t dW_t + \int_{\mathcal{E}} \psi_t(e)\widetilde{\pi}(de, dt) + dM_t, \tag{2.4}$$

where the main novelty is the generality of the filtration assumed (supporting at least a multidimensional Brownian motion as well as a Poisson random measure). Many of the assumptions and results of this thesis paper have been given in the previous chapter.

2.3 Continuity results for BSDE with non-Markovian singular terminal conditions

The case of non-Markovian singular terminal conditions was first studied in [37]. Here is the setup considered in this work: W is a single dimensional Brownian motion,

$$Y_s = Y_t + \int_s^t f(Y_r)dr + \int_s^t Z_r dW_r. \tag{2.5}$$

with terminal conditions

$$Y_T = \infty \cdot \mathbf{1}_{\{\tau \leq T\}} \tag{2.6}$$

and

$$Y_T = \infty \cdot \mathbf{1}_{\{\tau > T\}}, \tag{2.7}$$

where $\tau = \inf\{t > 0 : W_t \notin [a, b]\}$, and

$$f(y) = -y|y|^{q-1}. \tag{2.8}$$

Under this setting it was shown that there are solutions to the BSDE (2.5) if $p > 2$ for terminal value (2.6) and in scenario (2.7) if $p > 1$.

In [1], the above analysis is extended to a general driver and a filtration including a d-dimensional Brownian motion W and a Poisson random measure π (like the current thesis, [1] studies within the framework introduced in [21] and is summarized in the previous chapter); the hitting time τ in terminal condition (2.6) is generalized to any stopping time having a bounded density around the terminal time T and (2.7) is generalized to $Y_T = \infty \cdot \mathbf{1}_{A_T}$ where A_t is a decreasing sequence of events adapted to the filtration $\mathcal{F}_t$ and is continuous in probability at T. There are important connections between the analysis in the present thesis and the analysis in [1]; these connections are pointed out throughout the chapters that follows.

The work[27] study the same BSDE as in [21] and focuses on non-Markovian terminal conditions that are smooth functions of ω. The smoothness assumption allows this work to use functional Itô calculus to derive the continuity of the minimal super-solution of the BSDE.

2.4 BSDE with singular terminal values and control problems with constraints

Interestingly, BSDE with superlinear growth and singular terminal values reviewed above arise naturally in stochastic optimal control problems with terminal constraints modeling optimal liquidation of financial portfolios. The first paper in this connection is [3], which shows that the value function of the following stochastic optimal control problem can be represented by a BSDE with a singular terminal condition:

$$J(v(t)) = \mathbb{E}\left[\int_0^T \left(\alpha_u |v_u|^p + \beta_t |q_u|^p\right) du\right], q_s = q_t + \int_t^s v_u du, s > t, q_T = 0, \quad (2.9)$$

where α and β are $\mathbb{F}^W$ -progressively measurable non-negative stochastic processes and $p > 1$. The novel feature of this problem is the terminal constraint $q_T = 0$. They prove that the value function $\inf_v J_t(v)$ of the above control problem equals $q_t^p Y_t^{\min}$ where $Y^{\min}$ is the minimal supersolution of the BSDE

$$dY_t = \left((p-1)\frac{Y_t^q}{\alpha_t^{q-1}} - \beta_t\right) dt + Z_t dW_t, \quad (2.10)$$

where $1/q = 1 - 1/p$, and terminal condition $\lim_{t\to T} Y_t = \infty$; the terminal condition corresponds exactly to the constraint $q_T = 0$. They further proved that under integrability conditions:

1. $\alpha \in L^2(\Omega \times [0,T], \mathcal{P}, P \bigotimes \lambda)$ and $\frac{1}{\alpha^{q-1}} \in L^1(\Omega \times [0,T], \mathcal{P}, P \bigotimes \lambda)$
2. $\beta \in L^2(\Omega \times [0,t], \mathcal{P}, P \bigotimes \lambda)$ for all $t < T$ and $\mathbb{E}\int_0^T (T-s)^p \beta_s ds < \infty$,

BSDE (2.10) has a minimal super-solution $\langle Y, Z\rangle$. Furthermore, when α and β satisfy these conditions, the optimal control is given as

$$q(t) = \exp\left(-\int_0^t \left(\frac{Y_s}{\alpha_s}\right)^{q-1} ds\right). \quad (2.11)$$

The control problem (2.9) has the following finance interpretation: q is the position of a trader in an asset, the goal of the trader is to optimally close the position at time T (this corresponds to the constraint $q_T = 0$). The α term in the cost function represents transaction costs and the β term defines a measure of risk for the terminal wealth of the trader (see Chapter 6 and [16] for how these interpretations arise).

The work [22] shows that the above results can be generalized as follows: the q process is generalized to

$$W_t = w + \int_0^t \Lambda_r dr + \int_0^t \int_\Theta \Upsilon_r(\theta)\pi(d\theta, dr),$$

and the cost function is generalized to

$$J(W) = \mathbb{E}\left[\int_0^\tau \left(\alpha_u|\Lambda_u|^p + \beta_u|W_u|^p + \int_\Theta \aleph_s(\theta)|\Upsilon_u(\theta)|^p\mu(d\theta)\right) du + \xi|W_\tau|^p\right]$$

The components of this cost function are the primary market trading cost, the risk of having the position open, and slippage costs (cost of trading at jump times in the secondary market or dark pool see[19]). If τ is either an exit or a deterministic time, the BSDE associated with this cost process is of the form

$$dY_t = (p-1)\frac{Y_t^q}{\alpha_t^{q-1}}dt + F(t, Y_t, \Gamma_t)dt - \beta_t dt + \int_\Theta \Gamma(\theta)\tilde{\pi}(d\theta, dt) + dM_t. \quad (2.12)$$

The condition for the existence of a minimal supersolution are:

1. If the terminal time τ is deterministic, then there is a super-solution for (2.12) provided that $\alpha > 0$, $\beta \geq 0$ and for $l > 1$, $\mathbb{E}\left[\int_0^T (\alpha_t + (T-t)^p\beta_t)^l dt\right] < \infty$ and $\mathbb{E}\left[\int_0^T \frac{1}{\alpha^{q-1}}dt\right] < \infty$.

2. In the event τ is an exit time, if $\mathbb{E}[e^{\varphi\tau}] < \infty$ where $\exists\varphi > \mu(\Theta)$, $\exists K > \alpha_t > 0$ & $K > \beta$ and $\mathbb{E}\left[\int_0^n \frac{1}{\alpha^{q-1}}dt\right] + \mathbb{E}\left[\int_0^\tau \frac{1}{\alpha^{m(q-1)}}dt\right] < \infty$ then there is a super-solution for (2.12)

Under this backdrop,

$$W_u^* = w \exp\left[-\int_t^{u\wedge t}\left(\frac{Y_r}{\alpha_r}\right)^{q-1} dr\right] \exp\left[\int_t^{u\wedge t}\int_\Theta \ln(1-\varrho_r(\theta))\pi(d\theta, dr)\right]$$

is given as the optimal trading strategy where,

$$\varrho_r(\theta) = \frac{(Y_{u^-}+\Gamma_u(\theta))^{q-1}}{(Y_{u^-}+\Gamma_u(\theta))^{q-1}+\Upsilon_u(\theta)^{q-1}}.$$

In Chapter 6, we give an explicit solution to the stochastic optimal control problem (2.9) and the related BSDE when the terminal time of the problem is chosen to be $\tau_u = \inf\{t : W_t \geq u\}$, which corresponds to full liquidation as soon as the price process hits a target price u under the assumption that α is constant and $\beta = 0$.

CHAPTER 3

SOLVABLE STOPPING TIME AND MINIMAL SUPERSOLUTION

The next lemmas are useful consequences of the notion of solvable stopping times. First, note that the left-continuity assumption of $\mathbb{F}$ at time S is true for example if S is predictable and if $\mathbb{F}$ is a quasi-left continuous filtration: for any predictable stopping time τ, we have $\mathcal{F}_{\tau-} = \mathcal{F}_{\tau}$. This property of the filtration rules out the possibility that any of the involved processes has jumps at predictable, and a fortiori deterministic times. An important example is the filtration generated by the Brownian motion W and the orthogonal Poisson random measure π and S is given by (1.17).

Lemma 1. *Assume that S is solvable and suppose that the generator f satisfies Conditions* **(A)**. *Then, the BSDE* (1.1) *has a minimal supersolution on the time interval* $[\![0, S]\!]$ *with terminal condition $Y_S = \infty$.*

Proof. The arguments can be found in [22, Propositions 4 and 7]. The adaptation is straightforward in our setting since the arguments are not based on a particular form of the stopping time S. Only left-continuity of the filtration is important. □

Let us emphasize that Assumptions **(B)** are not necessary here, since solvability implies existence of a supersolution. In the rest of the paper we denote the minimal supersolution with terminal condition $+\infty$ a.s. at time S by $(Y^\infty, Z^\infty, \psi^\infty, M^\infty)$. Sometimes, if we want to stress the dependence w.r.t. S, $(Y^{S,\infty}, Z^{S,\infty}, \psi^{S,\infty}, M^{S,\infty})$ denotes it.

Lemma 2. *Assume that S is solvable and suppose that the generator f satisfies Conditions* **(A)***,* **(B2)***,* **(B3)** *and* **(B4)**. *Then, the BSDE* (1.1) *with a singular terminal value ξ at time S, has a minimal supersolution $(Y^{\min}, Z^{\min}, \psi^{\min}, M^{\min})$ on the time interval $[\![0, S]\!]$ with terminal condition $Y_S^{\min} = \xi$.*

Proof. Let us denote by $Y^{(k),\infty}$, the first component of the solution of the BSDE (1.1) with terminal condition k. Since S is solvable, and with **(A)**, $Y^{(k),\infty}$ is an increasing sequence converging to Y^∞.

Again from [22, Proposition 5], under **(A)**, **(B2)**, **(B3)** and **(B4)**, there exists a unique solution $(Y^{(k)}, Z^{(k)}, \psi^{(k)}, M^{(k)})$ to the BSDE (1.14) and (1.15). By comparison principle, a.s for any $t \geq 0$

$$Y_t^{(k)} \leq Y_t^{(k),\infty} \leq Y_t^\infty.$$

Hence we obtain an upper estimate on $Y^{(k)}$, independent of k, which replaces the upper bound (1.20). Arguing now as in [22], the existence of $(Y^{\min}, Z^{\min}, \psi^{\min}, M^{\min})$ is obtain. □

Note that the main result of Theorem 2 is the solvability of the first exit time S. The existence of $(Y^{\min}, Z^{\min}, \psi^{\min}, M^{\min})$ comes from the preceding lemma.

Before we move further, let us note the following:

Lemma 3. *Suppose a stopping time β is solvable. Suppose (Y, Z, ψ, M) is a supersolution of* (1.1) *with terminal condition ξ constructed as the limit of solutions with terminal condition $\xi \wedge L$. Then, the sequence β_n in Definition 2 can be chosen so that*

$$Y_t \leq n \text{ for } t < \beta_n. \tag{3.1}$$

Proof. Let $Y^{\beta,\infty}$ denote the first component of the supersolution for terminal condition ∞ and let $\beta_n^{1,\infty}$ be the sequence of β_n in Definition 2 for the same terminal condition. It follows from (1.12) and (1.2) that $Y^{\beta,\infty}$ has càdlàg sample paths on $[\![0, \beta]\!]$ and $\lim_{t\to\infty} Y^{\beta,\infty}_{t\wedge\beta} = \infty$. This implies that the hitting times

$$\beta_n^{2,\infty} \doteq \inf\{t : Y^{\infty,\beta}_{t\wedge\beta} \geq n\} \tag{3.2}$$

satisfy: $\beta_n^{2,\infty} \leq \beta$ and it is a non decreasing sequence. From the first property of a supersolution, this sequence converges almost surely to β. Now suppose that $\beta_N^{2,\infty} = \beta$ for some N (and thus for any $n \geq N$). It would mean that $Y^{\beta,\infty}$ has a jump at time β. In other words, the martingale parts have a jump at time β. But it is excluded in the definition 3. Thus

$$\beta_n^{2,\infty} \nearrow \beta \text{ as } n \nearrow \infty. \tag{3.3}$$

Then, if we replace the stopping times $\beta_n^{1,\infty}$ in Definition 2 with $\beta_n^{3,\infty} \doteq \beta_n^{1,\infty} \wedge \beta^{2,\infty}$ all of the conditions of the definition remain valid; furthermore

$$Y_t^{\beta,\infty} \leq n \text{ for } t < \beta_n^{3,\infty}, \tag{3.4}$$

holds. This proves the lemma for the terminal condition ∞. Let $Y^{\beta,L}$ denote the solution of (1.1) with terminal condition $Y_\beta = L$. Then, by definition $Y^{\beta,L}_{t\wedge\beta} \nearrow Y^{\beta,\infty}_{t\wedge\beta}$.

This and (3.4)

$$Y_t^{\beta,L} \le Y_t^{\beta,\infty} \le n \text{ for } t < \beta_n^{3,\infty}. \tag{3.5}$$

Let $Y^{\beta,\xi}$ be the supersolution of (1.1) with terminal condition $Y_\beta = \xi$. and let $Y^{\beta,\xi\wedge L}$ be the solution of (1.1) with terminal condition $Y_\beta = \xi \wedge L$. By the assumption of the lemma

$$Y_{t\wedge\beta}^{\beta,\xi\wedge L} \nearrow Y_{t\wedge\beta}^{\beta,\xi} \tag{3.6}$$

as $L \nearrow \infty$. By comparison principle for the solution of BSDE we have $Y_{t\wedge\beta}^{\beta,\xi\wedge L} \le Y_{t\wedge\beta}^{\beta,L}$. This, (3.5), (3.6), the definition (3.2) of $\beta_n^{2,\infty}$ and letting $L \nearrow \infty$ imply

$$Y_t^{\beta,\xi} \le Y_t^{\beta,\infty} \le n \text{ for } t < \beta_n^{3,\infty}. \tag{3.7}$$

Let $\beta_n^{1,\xi}$ be the sequence of stopping time appearing in the definition of the supersolution $Y^{\beta,\xi}$. Define $\beta_n^{2,\xi} \doteq \beta_n^{1,\xi} \wedge \beta_n^{3,\infty}$. From (3.1) and from the assumption that $\beta_n^{1,\xi} \nearrow \beta$ we infer $\beta_n^{2,\xi} \nearrow \beta$. This implies that if we replace $\beta_n^{1,\xi}$ with $\beta_n^{2,\xi}$, all of the conditions appearing in the definition of the supersolution $Y^{\xi,\beta}$ continue to hold; by (3.7) this sequence of stopping times also satisfy

$$Y_t^{L,\beta} \le Y_t^{\infty,\beta} \le n \text{ for } t < \beta_n^{2,\xi}. \tag{3.8}$$

This proves the lemma for the terminal condition ξ. □

If we work with the filtration $\mathbb{F}^W$ generated by the Brownian motion, then the BSDE (1.1) reduces to the following:

$$dY_t = -f(t, Y_t, Z_t)dt + Z_t dW_t. \tag{3.9}$$

Corollary 1. *In the Brownian filtration $\mathbb{F}^W$, if β is solvable, then* (3.1) *becomes:*

$$Y_t \le n \, for \, t \le \beta_n. \tag{3.10}$$

Proof. Since the trajectories of Y are now continuous, (3.1) can be strengthened to (3.10). □

Remark 4. *The estimate* (1.20) *implies that* $Y^{\min}$ *of* (1.16) *satisfies* $Y_t^{\min} \leq Cn^{2(p-1)}$ *almost surely if* $t \leq S_n$ *where* S_n *is as in* (1.21). *This property is a special case of the above corollary.*

CHAPTER 4

TERMINAL CONDITION ξ_1

In this chapter, we study terminal conditions of the form

$$\xi_1 = \infty \cdot \mathbf{1}_{\{\tau \leq S\}}$$

where τ is another stopping time. We know from [1, Section 2] that when $S = T$ is deterministic and τ has a bounded density around the terminal time T, the minimal supersolution of the BSDE (1.1) with terminal condition ξ_1 satisfies

$$\lim_{t \to T} Y_T^{\min} = \xi_1.$$

Our goal is to prove similar continuity results when S is a stopping time. For this we will consider two approaches: the first is an extension of the approach taken in [1, Section 2], section 4.1 focuses on this. We consider a new approach in the subsequent subsection 4.2.

4.1 First approach: The use of a Constructed Auxiliary Upperbound

The approach of [1, Section 2] can be summarized as follows:

1. Let Y^∞ be the minimal supersolution of (1.1) on the interval $[\![0, S]\!]$ with termi-

nal condition $Y_S = \infty$; define the auxilary terminal condition

$$\xi_1^{(\tau)} = \mathbf{1}_{\{\tau \leq S\}} Y_\tau^\infty.$$

2. Prove

$$\mathbb{E}\left[\left(\xi_1^{(\tau)}\right)^\varrho\right] < \infty \tag{4.1}$$

for some $\varrho > 1$, in particular, $\xi_1^{(\tau)}$ is not a singular terminal condition.

3. Let $\widehat{Y}^u$ be the solution of a linear BSDE with terminal condition $\xi_1^{(\tau)}$ whose driver term is chosen to guarantee $Y^{\min} \leq \widehat{Y}^u$ (the superscript u stands for upper bound).

4. Derive the continuity of $Y^{\min}$ from that of $\widehat{Y}^u$.

Let us emphasize that (4.1) implies that $\mathbb{P}(\tau = S) = 0$. Indeed if not, then

$$\mathbb{E}\left[\left(\xi_1^{(\tau)}\right)^\varrho\right] \geq \mathbb{E}\left[\mathbf{1}_{\{\tau = S\}} \left(Y_S^\infty\right)^\varrho\right] = +\infty.$$

Now we state the next result in the context of stopping times.

Theorem 3. *Assume that the stopping time S is solvable, such that Conditions* **(A)** *and* **(B)** *hold. Let τ be a stopping time such that there exists ϱ large enough (depending on δ and δ^* in* **(B3)***) such that* (4.1) *holds. Then $Y^{\min}$ is continuous at S, that is a.s.*

$$\lim_{t \to +\infty} Y_{t \wedge S}^{\min} = \xi_1.$$

Proof. We follow the scheme developed in [1]. Since S is solvable, there exists a minimal supersolution $(Y^\infty, Z^\infty, \psi^\infty, M^\infty)$ to the BSDE (1.1) with terminal condition $+\infty$ at time S.

First, we consider the (linear in y) generator

$$g(t, y, z, \psi) = \chi y + f(t, 0, z, \psi),$$

which satisfies all conditions **(A)**, and the terminal value $\xi_1^{(\tau)}$ at the random time S. Note that $\xi_1^{(\tau)}$ is $\mathcal{F}_{\tau \wedge S}$-measurable, thus $\mathcal{F}_S$-measurable. Let us check that (1.8) holds, namely for some $r > 1$ and $\rho > \nu(r)$

$$\mathbb{E}\left[e^{r\rho S}|\xi_1^{(\tau)}|^r + \int_0^S e^{r\rho t}|g(t, 0, 0, \mathbf{0})|^r dt\right] < +\infty.$$

Note that $g(t, 0, 0, \mathbf{0}) = f_t^0$ and **(B2)** holds. From the proof of [22, Proposition 5], using **(B3)**, there exists $r > 1$ and $\rho > \nu(r)$ such that $r\nu(r) < \delta$. Hence we can find $\gamma > 1$ such that $\mathbb{E}(e^{r\rho\gamma S}) < +\infty$. Hölder's inequality leads to:

$$\mathbb{E}\left[e^{r\rho S}|\xi_1^{(\tau)}|^r\right] \leq \left(\mathbb{E}e^{r\rho\gamma S}\right)^{1/\gamma}\left(\mathbb{E}|\xi_1^{(\tau)}|^{r\gamma*}\right)^{1/\gamma*}.$$

If $\varrho \geq r\gamma*$, then we deduce that $\mathbb{E}|\xi_1^{(\tau)}|^{r\gamma*} < +\infty$ and (1.8) is satisfied.

Then we have to verify that (1.11) holds for $\xi_1^{(\tau)}$. This can be done by linearizing g and using the same arguments as for (1.8). Applying Theorem 1 leads to the existence and the uniqueness of the solution $(\widehat{Y}^u, \widehat{Z}^u, \widehat{\psi}^u, \widehat{M}^u)$.

We next prove that $\widehat{Y}^u$ does serve as an upper bound on $Y^{(k)}$, the solution of the BSDE (1.1) with terminal condition $\xi_1 \wedge k = k\mathbf{1}_{\tau \leq S}$ at time S: a.s. for any $t \geq 0$

$$Y^{(k)}_{t\wedge\tau\wedge S} \leq \widehat{Y}^u_{t\wedge\tau\wedge S}.$$

Indeed by comparison principle, $Y^{(k)} \leq Y^\infty$. Hence, a.s. $Y^{(k)}_{\tau\wedge S} = Y^{(k)}_\tau \mathbf{1}_{\tau \leq S} \leq Y^\infty_\tau \mathbf{1}_{\tau\leq S} = \xi_1^{(\tau)}$. Since $f(t, y, z, \psi) \leq g(t, y, z, \psi)$ by Condition **(A1)**, we deduce the wanted result.

We conclude using some linearization procedure (see [1, Lemma 3]) that a.s. on the

$\mathcal{F}_S$-measurable set $\{\tau > S\}$, that

$$\lim_{t\to+\infty} \widehat{Y}^u_{t\wedge S} = 0.$$

Thereby a.s. on the same set

$$0 \leq \lim_{t\to+\infty} Y^{\min}_{t\wedge S} \leq \lim_{t\to+\infty} \widehat{Y}^u_{t\wedge S} = 0 = \xi_1.$$

The continuity is proved. □

Let us develop an example. Let us assume that S is the first exit time of Ξ given by (1.17), $S = S_D = \inf\{t \geq 0, \quad \Xi_t \notin D\}$, such that there exists a constant C such that (1.20) holds:

$$0 \leq Y^{\infty}_{t\wedge S} \leq \frac{C}{\text{dist}(\Xi_{t\wedge S})^{2(p-1)}}.$$

We also suppose that σ is uniformly elliptic:

$$\forall x \in \mathbb{R}^d, \quad \sigma\sigma^*(x) \geq \alpha \text{Id}, \tag{4.2}$$

which implies, for $\Xi_0 = x \in D$, Ξ_t has a density $\phi(t, x, \cdot)$ [13]. Under this assumption, to prove (4.1) it suffices to prove

$$\mathbb{E}\left[\mathbf{1}_{\{\tau\leq S\}} \frac{1}{\text{dist}(\Xi_\tau)^{\varrho 2(p-1)}}\right] < \infty, \tag{4.3}$$

for some $\varrho > 1$. Theorem 3 above gives:

$$\lim_{t\to\infty} Y^{\min}_{t\wedge S} = \xi_1,$$

assuming (4.3).

The expectation in (4.3) depends on the joint distribution of (τ, S, Ξ_S). We are not aware of results available in the current literature that would imply (4.3) under broad and general assumptions on these variables. A basic case that can be treated with techniques that we know of is when τ is independent of Ξ (and therefore of S). The next proposition proves (4.3) under this setting.

Proposition 1. *Suppose that S is the first exit time of Ξ given by* (1.17)*, that σ is uniformly elliptic, and that τ is independent of Ξ. If $q > 1 + 2\varrho$, then*

$$\mathbb{E}\left[\mathbf{1}_{\{\tau\leq S\}}\frac{1}{\text{dist}(\Xi_\tau)^{\varrho 2(p-1)}}\right] < \infty, \tag{4.4}$$

Proof. The equality $1/p + 1/q = 1$ and $q > 1 + 2\varrho$ imply $2(p-1)\varrho < 1$. Let us denote the distribution of τ by F_τ. The expectation (4.3) can then be written as

$$\mathbb{E}\left[\mathbf{1}_{\{\tau\leq S\}}\frac{1}{\text{dist}(\Xi_\tau)^{\varrho 2(p-1)}}\right] = \int_0^\infty \mathbb{E}\left[\mathbf{1}_{\{t\leq S\}}\frac{1}{\text{dist}(\Xi_t)^{\varrho 2(p-1)}}\right] dF_\tau(t).$$

Since S is the exit time of Ξ from a smooth domain with uniformly elliptic diffusion matrix, we have:

$$\mathbb{E}\left[\mathbf{1}_{\{\tau\leq S\}}\frac{1}{\text{dist}(\Xi_\tau)^{\varrho 2(p-1)}}\right] = \int_0^\infty \mathbb{E}\left[\mathbf{1}_{\{t< S\}}\frac{1}{\text{dist}(\Xi_t)^{\varrho 2(p-1)}}\right] dF_\tau(t)$$

that $\{\Xi_t \in D\} \supset \{t < S\}$ implies

$$\leq \int_0^\infty \mathbb{E}\left[\mathbf{1}_{\{\Xi_t\in D\}}\frac{1}{\text{dist}(\Xi_t)^{\varrho 2(p-1)}}\right] dF_\tau(t). \tag{4.5}$$

We next bound

$$\mathbb{E}\left[\mathbf{1}_{\{\Xi_t\in D\}}\frac{1}{\text{dist}(\Xi_t)^{\varrho 2(p-1)}}\right].$$

For $\Xi_0 = x \in D$, let $\phi(t, x, \cdot)$ be the density of Ξ_t. The expectation above then can be written as

$$\mathbb{E}\left[\mathbf{1}_{\{\Xi_t\in D\}}\frac{1}{\text{dist}(\Xi_t)^{\varrho 2(p-1)}}\right] = \int_D \phi(t, x, y)\frac{1}{\text{dist}(y)^{\varrho 2(p-1)}}dy. \tag{4.6}$$

Define $D_\epsilon = \{x \in D : \text{dist}(x) \leq \epsilon\}$ for $\epsilon > 0$; by [14, Lemma 14.16] there exists $\epsilon_1' > 0$ such that dist is C^2 in $D_{\epsilon_1'}$. Therefore one can choose $\epsilon_1 \in (0, \epsilon_1']$ so that dist is smooth on D_{ϵ_1} and $x \notin D_{\epsilon_1}$. The continuity of dist implies that D_ϵ is closed; D_ϵ is therefore compact since $D_\epsilon \subset D$ and D is bounded. This, the continuity of dist and $x \notin D_{\epsilon_1}$ imply

$$C_1 \doteq \inf_{y \in D_{\epsilon_1}} |x - y| > 0. \tag{4.7}$$

Since b and σ are Lipschitz continuous and since σ is uniformly elliptic, from [13, page 16] we have the following Aronson's estimate on $\phi(t,x,y)$:

$$\phi(t,x,y) \leq \frac{C_2}{t^{d/2}} e^{-\frac{\lambda_0 |y-x|^2}{4t}}.$$

This and (4.7) imply

$$\phi(t,x,y) \leq \frac{C_2}{t^{d/2}} e^{-\frac{\lambda_0 C_1^2}{4t}},$$

for $y \in D_{\epsilon_1}$. The right side of this inequality is continuous and bounded for $t \in [0,\infty]$. Therefore,

$$C_3 \doteq \sup_{t\in[0,\infty], y\in D_{\epsilon_1}} \phi(t,x,y) \leq \sup_{t\in[0,\infty], y\in D_{\epsilon_1}} \frac{C_2}{t^{d/2}} e^{-\frac{\lambda_0 C_1^2}{4t}} < \infty. \tag{4.8}$$

We now decompose (4.6) into two integrals over D_{ϵ_1} and $D \setminus D_{\epsilon_1}$:

$$\begin{aligned} \mathbb{E}\left[\mathbf{1}_{\{\Xi_t \in D\}} \frac{1}{\text{dist}(\Xi_t)^{\varrho 2(p-1)}}\right] &= \int_D \phi(t,x,y) \frac{1}{\text{dist}(y)^{\varrho 2(p-1)}} dy \\ = \int_{D\setminus D_{\epsilon_1}} \phi(t,x,y) \frac{1}{\text{dist}(y)^{\varrho 2(p-1)}} dy. &+ \int_{D_{\epsilon_1}} \phi(t,x,y) \frac{1}{\text{dist}(y)^{\rho 2(p-1)}} dy \\ \leq \frac{1}{\epsilon_1^{2\varrho(p-1)}} + \int_{D_{\epsilon_1}} \phi(t,x,y) \frac{1}{\text{dist}(y)^{\varrho 2(p-1)}} dy. \end{aligned} \tag{4.9}$$

the last inequality coming from: $\text{dist}(y) > \epsilon_1$ for $y \in D \setminus D_{\epsilon_1}$.

It remains to bound the last integral. For this note that $\text{dist}(\cdot)$ is C^2 over D_{ϵ_1}. Furthermore, ∂D is the 0-level curve of $\text{dist}(\cdot)$, in particular, for $y \in \partial D$, the gradient $\nabla\text{dist}(y)$ is normal to ∂D. ∂D is a C^1 surface, with nonvanishing normal at everypoint. It follows from these and the definition of $\text{dist}(\cdot)$ that $\nabla\text{dist}(\cdot)$ satisfies

$|\nabla \text{dist}(y)| = 1$ for $y \in \partial D$. Now define

$$E_\epsilon = \{y \in D : \text{dist}(y) > \epsilon\} = D \setminus D_\varepsilon.$$

Since dist$(\cdot)$ is $C^2(D_{\epsilon_1})$ implies that ∂D_{ϵ_1} is a C^2 bounded surface and that the function

$$A(\epsilon) = \text{Area}(\partial E_\epsilon)$$

is C^1 over the interval $[0, \epsilon_1]$. In particular, it is continuous and satisfies

$$C_4 \doteq \sup_{\epsilon \in [0,\epsilon_1]} A(\epsilon) < \infty. \tag{4.10}$$

This and the definition of dist$(\cdot)$ imply $|\nabla \text{dist}(y)| = 1$ for $y \in \partial D_\epsilon$ for $\epsilon \leq \epsilon_1$. We are now in a setting where we can apply the co-area formula [11, Theorem 5, page 713], which gives

$$\int_{D_{\epsilon_1}} \phi(t,x,y) \frac{1}{\text{dist}(y)^{\varrho 2(p-1)}} dy = \int_0^{\epsilon_1} \left(\int_{\partial E_\epsilon} \phi(t,x,y) dS \right) \frac{1}{\epsilon^{\varrho 2(p-1)}} d\epsilon.$$

$\partial E_\epsilon \subset D_{\epsilon_1}$ and (4.8) imply

$$\leq \int_0^{\epsilon_1} \left(\int_{\partial E_\epsilon} C_3 dS \right) \frac{1}{\epsilon^{\varrho 2(p-1)}} d\epsilon.$$

This and (4.10) give

$$\leq C_3 C_4 \int_0^{\epsilon_1} \frac{1}{\epsilon^{\varrho 2(p-1)}} d\epsilon.$$

Recall that $\varrho 2(p-1) < 1$. This and the last line imply

$$\int_{D_{\epsilon_1}} \phi(t,x,y) \frac{1}{\text{dist}(y)^{\varrho 2(p-1)}} dy < C_5, \tag{4.11}$$

where

$$C_5 \doteq C_3 C_4 \int_0^{\epsilon_1} \frac{1}{\epsilon^{\varrho 2(p-1)}} d\epsilon < \infty.$$

The bound (4.11) we have just derived and (4.9) imply

$$\mathbb{E}\left[\mathbf{1}_{\{\Xi_t \in D\}} \frac{1}{\mathrm{dist}(\Xi_t)^{\varrho 2(p-1)}}\right] \leq \frac{1}{\epsilon_1^{\varrho 2(p-1)}} + C_5.$$

This and (4.5) imply (4.4). □

For example, if f only depends on y and is non increasing ($\chi = 0$), then it is sufficient to have $q > 3$.

4.2 A new argument for ξ_1 based on solvability

In the rest of the paper, to clearly state the ideas and for a less technical presentation, we will restrict our attention to the Brownian framework, i.e., we assume that $\mathbb{F} = \mathbb{F}^W$ is the filtration generated by the d-dimensional Brownian motion W. Therefore, (1.1) reduces to (3.9), that is:

$$dY_t = -f(t, Y_t, Z_t)dt + Z_t dW_t.$$

The continuity arguments in Section 4.1 above and in [1, Section 2] use the solution of a linear auxiliary BSDE as an upper bound to the minimal supersolution. In this section, we would like to explore a new upper bound that is based directly on the original nonlinear BSDE. As will be seen, whenever applicable, this is more natural and leads to less strict conditions on the parameter q of Condition **(B1)**. We assume τ and S to be solvable in the sense of Definition 3. Let $Y^{S,\infty}$ and $Y^{\tau,\infty}$ denote the ∞-supersolutions[1] corresponding to τ and S. The main idea of the present section as compared to that of Section 4.1 and [1, Section 2] is the following: we replace the upper bound process $\widehat{Y}^u$ of the proof of Theorem 3 with $Y^{\tau,\infty}$.

[1] When we refer to Y as the solution, we mean the first component Y of a solution (Y, Z).

Theorem 4. *Suppose τ and S are solvable in the sense of Definition 3. Then a supersolution $Y^{\min}$ of* (3.9) *with terminal condition $Y_S^{\min} = +\infty \cdot \mathbf{1}_{\{\tau \le S\}}$ exists and*

$$\lim_{t \to \infty} Y_{t \wedge S}^{\min} = +\infty \cdot \mathbf{1}_{\{\tau \le S\}} = \xi_1. \tag{4.12}$$

Proof. By assumptions, there exists a supersolution $Y^{S,\infty}$ to the BSDE with terminal condition $Y_S = \infty$ and this supersolution is the limit of processes $Y^{(L)}$ which are solutions of the same BSDE with terminal condition $Y_S = L$. Let $\xi \ge 0$ be an arbitrary terminal condition and let $Y^{(L),\xi}$ be the solution of (3.9) with terminal condition $Y_S = \xi \wedge L$. Comparison with $Y^{(L)}$ imply that $\lim_{L \nearrow \infty} Y^{(L)}$ defines the minimal supersolution $Y^{\min}$ to (3.9) with terminal condition ξ. By assumption τ is solvable, there exists a process $Y^{\tau,\infty}$ that is a supersolution to the BSDE (3.9) with terminal condition $Y_\tau = \infty$. Let τ_n be the sequence of increasing stopping times in Definition 2 associated with this supersolution and let $Y^{\tau,\infty,L}$ be the sequence of solutions of (3.9) with terminal condition $Y_\tau = L$; by definition

$$Y^{\tau,\infty} = \lim_{L \nearrow \infty} Y^{\tau,\infty,L}.$$

By Corollary 1, $Y^{\tau,\infty}$ is bounded by n in the interval $[\![0, \tau_n]\!]$.

Similarly, let $Y^{S,\xi_1,L}$ be the sequence of solutions of the BSDE (3.9) with terminal condition $Y_S = \xi_1 \wedge L = L \cdot \mathbf{1}_{\{\tau \le S\}.}$. We will now prove

$$Y_t^{S,\xi_1,L} \le Y_t^{\tau,\infty}, \quad t \le \tau_n \wedge S. \tag{4.13}$$

To prove this consider, for $L_1 > 0$ the solution Y^{S,ξ_1,L,L_1} of the BSDE (3.9) with terminal condition $Y_{\tau \wedge S} = \left(Y_\tau^{S,\xi_1,L} \mathbf{1}_{\{\tau \le S\}}\right) \wedge L_1 = (Y_\tau^{S,\xi_1,L} \wedge L_1) \mathbf{1}_{\{\tau \le S\}}$, which is $\mathcal{F}_{\tau \wedge S}$-measurable. We will compare this process with Y^{τ,L_1}, the solution of (3.9) with terminal condition $Y_\tau = L_1$, on the time interval $[\![0, \tau \wedge S]\!]$. By its definition, the terminal value of Y^{S,ξ_1,L,L_1} at time $\tau \wedge S$ equals,

$$Y^{S,\xi_1,L,L_1}_{\tau\wedge S} = (Y^{S,\xi_1,L}_{\tau} \wedge L_1)\mathbf{1}_{\{\tau \le S\}}$$

which is bounded by

$$L_1 \mathbf{1}\{\tau \le S\}. \tag{4.14}$$

Again by definition

$$\begin{aligned} Y^{\tau,L_1}_{\tau\wedge S} &= Y^{\tau,L_1}_{\tau}\mathbf{1}_{\{\tau\le S\}} + Y^{\tau,L_1}_{S}\mathbf{1}_{\{S<\tau\}} \\ &= L_1 + Y^{\tau,L_1}_{S}\mathbf{1}_{\{S<\tau\}}. \end{aligned}$$

It follows from this $Y^{\tau,L_1} \ge 0$ and (4.14) that

$$Y^{S,\xi_1,L,L_1}_{\tau\wedge S} \le Y^{\tau,L_1}_{\tau\wedge S}. \tag{4.15}$$

The processes Y^{S,ξ_1,L,L_1} and Y^{τ,L_1} are solutions of the (3.9) on the interval $[\![0,\tau\wedge S]\!]$ (in the sense of Theorem 1). This fact, $\tau_n \wedge S \le \tau \wedge S$, the inequality (4.15) and the comparison principle for BSDE imply

$$Y^{S,\xi_1,L,L_1}_t \le Y^{\tau,L_1}_t, \text{ for } t \in [\![0,\tau_n\wedge S]\!].$$

Letting $L_1 \nearrow \infty$ gives (4.13). Recall that $Y^{\tau,\infty}$ is bounded by n in the interval $[\![0,\tau_n]\!]$. This and (4.13) implies the same bound for $Y^{S,\xi_1,L}$. Letting $L \nearrow \infty$ we discover that the process Y^{S,ξ_1} is a solution of (3.9) in the interval $[\![0,\tau_n\wedge S]\!]$ with terminal condition

$$\xi_1\mathbf{1}_{\{S<\tau_n\}} + Y^{S,\xi_1}_{\tau_n}\mathbf{1}_{\{\tau_n\le S\}} = Y^{S,\xi_1}_{\tau_n}\mathbf{1}_{\{\tau_n\le S\}} \le n.$$

In particular, Y^{S,ξ_1} is continuous on $[\![0,\tau_n\wedge S]\!]$ and satisfies

$$\lim_{t\to\infty} Y^{S,\xi_1}_{t\wedge\tau_n\wedge S} = Y^{S,\xi_1}_{\tau_n\wedge S} - \Delta Y^{S,\xi_1}_{\tau_n\wedge S}.$$

Now over the event $\{\tau_n > S\}$, $Y^{S,\xi_1}_{\tau_n \wedge S} = 0$, and since the filtration is continuous at time S, there is no jump at time S. Thus over the event $\{\tau_n > S\}$

$$\lim_{t\to\infty} Y^{S,\xi_1}_{t\wedge\tau_n\wedge S} = 0.$$

Since $Y^{S,\xi_1} = Y^{\min}$, and

$$\bigcup_{n=1}^{\infty} \{\tau_n > S\} = \{\tau > S\}$$

implies (4.12). □

4.3 An example in one space dimension

In this section we go back to the setup studied in [37, Section 2]: the driver is deterministic and only a function of y:

$$f(y) = -y|y|^{q-1},$$

the terminal time S is deterministic T and the terminal condition is

$$Y_T = \infty \cdot \mathbf{1}_{\{\tau \le T\}}, \tag{4.16}$$

where τ is the first exit time of W from the interval $(0, L)$. Note that, since f is deterministic and since the terminal conditions only depend on W, the solution (Y, Z, ψ, M) of the BSDE (1.1) is reduced to $(Y, Z, 0, 0)$ and the BSDE can be reduced to:

$$Y_s = Y_t + \int_s^t f(Y_r)dr + \int_s^t Z_r dW_r. \tag{4.17}$$

Theorem 2.1 of [37] states that for $q > 2$ the minimal supersolution of the BSDE (4.17) with terminal condition (4.16) is continuous at time T. Let y_t denote the solution of $\frac{dy}{dt} = -f(y)$ on the interval $[0, T]$ with terminal value $y_T = \infty$, i.e.,

$$y_t \doteq ((q-1)(T-t))^{1-p}, \quad t < T, \quad 1/p + 1/q = 1. \tag{4.18}$$

The proof of [37, Theorem 2.1] is based on the following integrabilty result:

$$\mathbb{E}[y_\tau \mathbf{1}_{\{\tau \le T\}}] = \mathbb{E}[y_\tau \mathbf{1}_{\{\tau < T\}}] < \infty. \tag{4.19}$$

As in the proof of Theorem 3, [37] constructs a linear process that is continuous at time T to find a continuous upperbound on the minimal supersolution (which implies the continuity of the minimal supersolution); the bound (4.19) ensures that the upper bound linear process is well defined. The bound (4.19) requires $q > 2$ and that is the reason why this was assumed in [37] in its treatment of the terminal condition (4.16). We will now derive the same continuity result under the assumption $q > 1$ using Theorem 4 above.

To apply Theorem 4 to the present setup we need T and τ to be solvable. This essentially means that the BSDE has supersolutions with terminal value ∞ at these terminal times. The supersolution for terminal time T is the deterministic process $t \mapsto y_t$. The fact that τ is solvable can be derived from (1.20). Instead of invoking this general result, in the following lemma we will make use of the simple nature of f and W to explicitly construct the supersolution $Y^{\tau,\infty}$ with terminal condition $Y_\tau = \infty$. Following [33, page 307] we will use

$$\boldsymbol{x}(v, v_l) \doteq v_l^{1-\frac{q+1}{2}} \left(\frac{q+1}{4} \right)^{1/2} \int_1^{v/v_l} \left(u^{q+1} - 1 \right)^{-1/2} du. \tag{4.20}$$

to construct solutions to the ODE

$$\frac{1}{2} V_{xx} - V^q = 0. \tag{4.21}$$

The function $\boldsymbol{x}$ is strictly increasing in v, furthermore, $q > 1$ implies $\boldsymbol{x}(\infty, v_l) < \infty$. Define

$$\boldsymbol{L}(v_l) = \boldsymbol{x}(\infty, v_l).$$

Let $\boldsymbol{x}^{-1}(\cdot, v_l)$ denote the inverse of $\boldsymbol{x}(\cdot, v_l)$. Now define

$$\boldsymbol{v}(x, v_l) \doteq \boldsymbol{x}^{-1}(|x - L/2|, v_l).$$

Lemma 4. *On the interval* $[L/2 - \boldsymbol{L}(v_l), L/2 + \boldsymbol{L}(v_l)]$, $\boldsymbol{v}(\cdot, v_l)$ *satisfies* (4.21) *with boundary conditions* ∞ *on both sides.*

Proof. If $g(u)$ is the inverse function of $f(x)$, then we have, $g(f(x)) = x$. By differentiating both sides we derive:

$$g'(f(x))f'(x) = 1 \iff f'(x) = \frac{1}{g'(f(x))}.$$

Using this identity we have:

$$f''(x) = \frac{d}{dx}f'(x) = \frac{d}{dx}\left(\frac{1}{g'(f(x))}\right) = -\frac{g''(f(x))f'(x)}{g'(f(x))^2} = -\frac{g''(f(x))}{g'(f(x))^3} = -\frac{g''(u)}{g'(u)^3}.$$

Therefore:

$$\boldsymbol{v}_{xx} = -\frac{\boldsymbol{x}_{vv}}{\boldsymbol{x}_v^3} = 2\frac{\boldsymbol{v}^q}{v_l}.$$

By letting $v_l = 1$, we have $\frac{1}{2}\boldsymbol{v}_{xx} = \boldsymbol{v}^q$. Furthermore, since $[L/2 - \boldsymbol{L}(v_l), L/2 + \boldsymbol{L}(v_l)] \subseteq \mathcal{R}(\boldsymbol{x})$, then $[L/2 - \boldsymbol{L}(v_l), L/2 + \boldsymbol{L}(v_l)] \subseteq \mathcal{D}(\boldsymbol{v})$. Since $\boldsymbol{L}(l) = \boldsymbol{x}(\infty, l)$ we have the boundary conditions ∞ satisfied.

□

To construct a supersolution of (4.17), we want to solve (4.21) in the interval $[0, L]$ with ∞ terminal conditions. Note that $\boldsymbol{L}(0) = \infty$ and $\boldsymbol{L}(\infty) = 0$ and $\boldsymbol{L}$ is a decreasing smooth function. It follows that there is a unique v^* such that $\boldsymbol{L}(v^*) = L/2$. Then for $v_l = v^*$, $\boldsymbol{v}(x, v^*)$ solves (4.21) in the interval $[0, L]$ with ∞ terminal conditions. For our argument we also need solutions to (4.21) in the time interval $[0, L]$

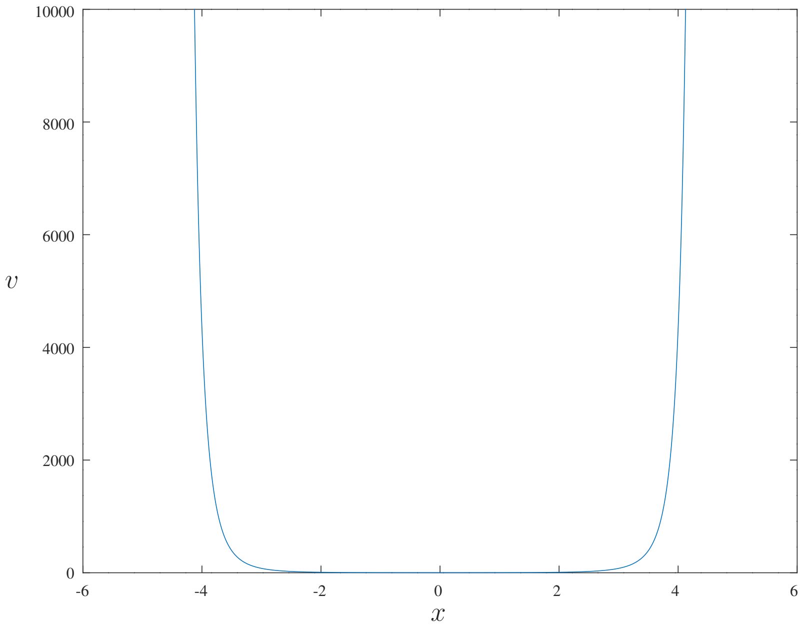

Figure 4.1: Solution for ODE 4.21 when $v_1 = 1$, and $q = 1.4$

with boundary condition n on both sides. For this purpose, the next lemma constructs a sequence $0 < v_n \nearrow v^*$ such that $\boldsymbol{x}(n, v_n) = L/2$.

Lemma 5. *There exists a sequence $0 < v_n \nearrow v^*$ such that $\boldsymbol{x}(n, v_n) = L/2$.*

Proof. Recall that v^* is the unique solution of $\boldsymbol{x}(\infty, v^*) = L/2$, i.e.,

$$(v^*)^{1-\frac{q+1}{2}} \left(\frac{q+1}{4}\right)^{1/2} \int_1^\infty \left(u^{q+1} - 1\right)^{-1/2} du = L/2.$$

This implies in particular

$$\boldsymbol{x}(1, v^*) = (v^*)^{1-\frac{q+1}{2}} \left(\frac{q+1}{4}\right)^{1/2} \int_1^{1/v^*} \left(u^{q+1} - 1\right)^{-1/2} du < L/2.$$

Furthermore, the function $v_l \mapsto \boldsymbol{x}(1, v_l)$ is continuous on $(0, v^*]$ and increases to ∞ as $v_l \searrow 0$. This implies that there exists $v_1 < v^*$ satisfying $\boldsymbol{x}(1, v_1) = L/2$. Now note $\boldsymbol{x}(2, v_1) > L/2$ and $\boldsymbol{x}(2, v^*) < L/2$. Applying the same argument gives $v_2 \in (v_1, v^*)$ satisfying $\boldsymbol{x}(2, v_2) = L/2$. Repeating the same argument inductively gives us an increasing sequence v_n bounded by v^* solving $\boldsymbol{x}(n, v_n) = L/2$. The limit v^{**} of this sequence satisfies $\boldsymbol{x}(\infty, v^{**}) = L/2$. Recall that v^* is the unique solution of this equation. This yields $v_n \nearrow v^*$. ☐

We can now state and prove the generalization of [37, Theorem 4] to $q > 1$:

Theorem 5. *For $q > 1$ the minimal supersolution of* (4.17) *with terminal condition $Y_T = \infty \cdot \mathbf{1}_{\{\tau \leq T\}}$ is continuous at time T.*

Proof. By the previous lemma there exists $v_n \nearrow v^*$ that solves $\boldsymbol{x}(n, v_n) = L/2$. It follows from this and Lemma 4 that $\boldsymbol{v}(\cdot, v_n)$ solves (4.21) on $[0, L]$ with terminal condition n on both sides and that $\boldsymbol{v}(\cdot, v_n) \to \boldsymbol{v}(\cdot, v^*)$. The comparison principle for the equation (4.21) implies that in fact $\boldsymbol{v}(\cdot, v_n) \nearrow \boldsymbol{v}(\cdot, v^*)$. Now define the processes

$$Y_t^{\tau,n} = \boldsymbol{v}(W_t, v_n), Y_t^{\tau,\infty} = \boldsymbol{v}(W_t, v^*).$$

(3.9) Itô's formula implies that $Y_t^{\tau,n}$ solves (4.17) with terminal condition $Y_\tau = n$. Define τ_n to be the first time W hits $[1/n, L - 1/n]$. Itô's formula implies $Y^{\tau,\infty}$ satisfies (1.12) (with $\beta_n = \tau_n$) and the definition of $\boldsymbol{v}(\cdot, v^*)$ and the continuity of the sample paths of W imply (1.2) with $\xi = \infty$. Therefore, $Y^{\tau,\infty}$ is a supersolution of (4.17) with terminal condition $Y_\tau = \infty$. Furthermore, $\boldsymbol{v}(\cdot, v_n) \nearrow \boldsymbol{v}(\cdot, v^*)$ implies $Y_t^{\tau,n} \nearrow Y_t^{\tau,\infty}$. These imply that τ satisfies all of the conditions of being solvable. T is also solvable because it is deterministic. Theorem 4 now implies the statement of the present theorem. ☐

4.4 A numerical example

From The proof of 4 we have the following identities:

$$\boldsymbol{v}_x = 2\left(\frac{(\boldsymbol{v}^{q+1}-1)}{q+1}\right)^{1/2} \quad \text{and} \quad \boldsymbol{v}_{xx} = 2\boldsymbol{v}^q.$$

Let $dY_t = \sigma dW_t$. and apply Itô formula to $\boldsymbol{v}(x,1)$, we derive:

$$\boldsymbol{v}(Y_t) = \xi + \int_t^{T\wedge\tau} \sigma^2 \boldsymbol{v}^q(Y_s)ds + \int_t^{T\wedge\tau} 2\sigma \left(\frac{(\boldsymbol{v}^{q+1}(Y_s)-1)}{q+1}\right)^{1/2} dW_s.$$

For a discrete model,where $dY_t = \sqrt{\Delta t}dW_t$, We have:

$$v(Y_n) = \xi + \sum_{s=1}^{n\wedge\tau} v^q(Y_{s-1})\Delta s^2 + 2\sum_{s=1}^{n\wedge\tau} \sqrt{\Delta s}\left(\frac{(v^{q+1}(Y_{s-1})-1)}{q+1}\right)^{1/2} dW_s.$$

Further, we assumed that $\Delta t = 1/10000$, $q = 1.4$, $\xi = 5$ This results in $\pm L = \pm 3.5307$. The Graphs below models Y_t When $\tau < S$ and $\tau \geq S$ (here $S = T$ is deterministic time).

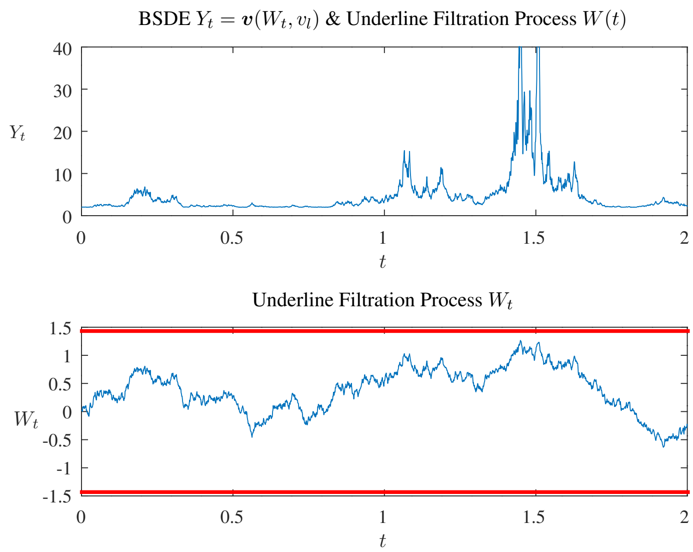

Figure 4.2: Simulation 1: The W process remains within the $\pm L$ band.

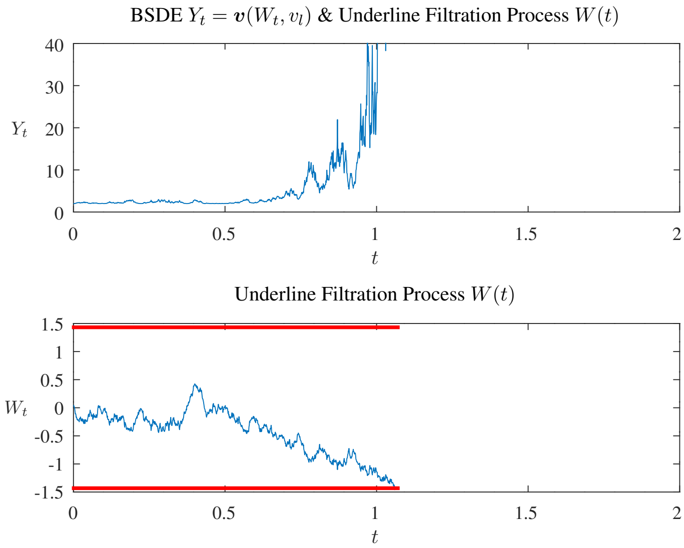

Figure 4.3: Simulation 2: The W process exits the $\pm L$ band and terminates the at time τ.

CHAPTER 5

TERMINAL CONDITION ξ_2

In this chapter, we examine the terminal condition $\xi_2 = \infty \cdot \mathbf{1}_{\{\tau > S\}}$ by assuming S is solvable. This means that there exists a minimal supersolution $Y^{S,\infty} \geq 0$ to (3.9) with terminal condition $Y^{S,\infty}_S = \infty$ and a sequence of stopping times $S_n \nearrow S$ such that $Y^{S,\infty}_t \leq n$ for $t \leq S_n$. (Definitions 2 and 3, Lemma 1 and Corollary 1).

Our continuity result is as follows:

Theorem 6. *Suppose S is solvable and τ is an arbitrary stopping time such that $\mathbb{P}(S = \tau) = 0$. Then the BSDE* (3.9) *has a supersolution in the time interval $[\![0, S]\!]$ with terminal condition $Y_S = \xi_2 = \infty \cdot \mathbf{1}_{\{\tau > S\}}$. Furthermore this supersolution is continuous at S:*

$$\lim_{t \to \infty} Y_{S \wedge t} = \xi_2. \tag{5.1}$$

This generalizes [1, Theorem 2] which assumes deterministic terminal times, to random terminal times. The main idea of the proof of [1, Theorem 2] generalized to the current setup is as follows: we construct a sequence of supersolutions to (3.9) with terminal conditions $Y_S = \infty \cdot \mathbf{1}_{\{\tau > S_n\}}$ where S_n is the sequence of stopping times approximating S. Note that these processes are all defined over the time interval $[\![0, S]\!]$, $S_n < S$ allows one to prove they are all continuous at time S. This, $\infty \cdot \mathbf{1}_{\{\tau > S_n\}} \geq \infty \cdot \mathbf{1}_{\{\tau > S\}}$ and comparison principle for BSDE allow one to argue that Y^{S,ξ_2} is also continuous at S, which is the result we seek.

Let us define several processes that will be useful in the proof of Theorem 6, as solution of the BSDE (3.9) over the time interval $[\![0, S]\!]$, changing the terminal condition at time S:

- $Y^{S,L}$ corresponds to the terminal condition L ;
- $Y^{S,0}$ to the terminal condition 0 ;
- $Y^{S,\xi_2,L,n}$ to the terminal condition $L \cdot \mathbf{1}_{\{\tau > S_n\}}$.

Note that these terminal conditions are $\mathcal{F}_S$-measurable and bounded. Hence from Theorem 1 and the conditions **(B)**, these solutions are well defined and unique (in the sense of Definition 1).

Let $Y^{S_n,\xi_2,L}$ be the solution of (3.9) in the time interval $[\![0, S_n]\!]$ with terminal condition

$$Y_{S_n} = Y^{S,L}_{S_n} \cdot \mathbf{1}_{\{\tau > S_n\}} + Y^{S,0}_{S_n} \cdot \mathbf{1}_{\{\tau \leq S_n\}}.$$

The existence and uniqueness of $Y^{S_n,\xi_2,L}$ comes from the estimates on $Y^{S,L}$ and $Y^{S,0}$ in Theorem 1. We begin our argument with the following lemma.

Lemma 6. *The process $Y^{S,\xi_2,L,n}$ has the following structure:*

$$Y^{S,\xi_2,L,n}_t = Y^{S_n,\xi_2,L}_t \mathbf{1}_{t \leq S_n} + Y^{S,0}_t \cdot \mathbf{1}_{t > S_n} \cdot \mathbf{1}_{\tau \leq S_n} + Y^{S,L}_t \cdot \mathbf{1}_{t > S_n} \cdot \mathbf{1}_{\tau > S_n}. \qquad (5.2)$$

Proof. First, $S_n < S$ implies that the right side of (5.2) defines an adapted and continuous process, denoted by $\mathcal{Y}$, with bounded terminal condition $Y_S = L \cdot \mathbf{1}_{\{\tau > S_n\}}$.

Let us show that $\mathcal{Y}$ satisfies also (3.9). We define similarly:

$$\mathcal{Z}_t = Z_t^{S_n,\xi_2,L}\mathbf{1}_{t\leq S_n} + Z_t^{S,0}\cdot\mathbf{1}_{t>S_n}\cdot\mathbf{1}_{\tau\leq S_n} + Z_t^{S,L}\cdot\mathbf{1}_{t>S_n}\cdot\mathbf{1}_{\tau>S_n}.$$

For any $0\leq t\leq T$, let us distinguish several cases:

- If $0\leq t\leq T\leq S_n<S$, then since $Y^{S_n,\xi_2,L}$ solves (3.9) on $[\![0,S_n]\!]$:

$$\begin{aligned}\mathcal{Y}_{t\wedge S} = Y_t^{S_n,\xi_2,L} &= Y_T^{S_n,\xi_2,L} + \int_t^T f(u,Y_u^{S_n,\xi_2,L},Z_u^{S_n,\xi_2,L})du - \int_t^T Z_u^{S_n,\xi_2,L}dW_u\\ &= \mathcal{Y}_{T\wedge S} + \int_{t\wedge S}^{T\wedge S} f(u,\mathcal{Y}_u,Z_u^{S_n,\xi_2,L})du - \int_{t\wedge S}^{T\wedge S} Z_u^{S_n,\xi_2,L}dW_u.\end{aligned}$$

- If $S_n<t\leq T$, then

$$\begin{aligned}\mathcal{Y}_{t\wedge S} &= Y_{t\wedge S}^{S,0}\cdot\mathbf{1}_{\tau\leq S_n} + Y_{t\wedge S}^{S,L}\cdot\mathbf{1}_{\tau>S_n}\\ &= \mathcal{Y}_{T\wedge S} + \int_{t\wedge S}^{T\wedge S}\left[f(u,Y_u^{S,0},Z_u^{S,0})\cdot\mathbf{1}_{\tau\leq S_n} + f(u,Y_u^{S,L},Z_u^{S,L})\cdot\mathbf{1}_{\tau>S_n}\right]du\\ &- \int_{t\wedge S}^{T\wedge S}\left[Z_u^{S,0}\cdot\mathbf{1}_{\tau\leq S_n} + Z_u^{S,L}\cdot\mathbf{1}_{\tau>S_n}\right]dW_u\\ &= \mathcal{Y}_{T\wedge S} + \int_{t\wedge S}^{T\wedge S} f(u,\mathcal{Y}_u,Z_u^{S,0}\cdot\mathbf{1}_{\tau\leq S_n} + Z_u^{S,L}\cdot\mathbf{1}_{\tau>S_n})du\\ &- \int_{t\wedge S}^{T\wedge S}\left[Z_u^{S,0}\cdot\mathbf{1}_{\tau\leq S_n} + Z_u^{S,L}\cdot\mathbf{1}_{\tau>S_n}\right]dW_u\end{aligned}$$

  since both sets $\{\tau\leq S_n\}$ and $\{\tau<S_n\}$ are $\mathcal{F}_{S_n}$-measurable.

- If $0\leq t\leq S_n<T$, then

$$\begin{aligned}\mathcal{Y}_{t\wedge S} = Y_t^{S_n,\xi_2,L} &= \mathcal{Y}_{S_n} + \int_{t\wedge S}^{S_n} f(u,\mathcal{Y}_u,Z_u^{S_n,\xi_2,L})du - \int_{t\wedge S}^{S_n} Z_u^{S_n,\xi_2,L}dW_u\\ &= Y_{S_n}^{S,0}\cdot\mathbf{1}_{\tau\leq S_n} + Y_{S_n}^{S,L}\cdot\mathbf{1}_{\tau>S_n}\\ &+ \int_{t\wedge S}^{S_n} f(u,\mathcal{Y}_u,Z_u^{S_n,\xi_2,L})du - \int_{t\wedge S}^{S_n} Z_u^{S_n,\xi_2,L}dW_u\\ &= \mathcal{Y}_{T\wedge S} + \int_{S_n}^{T\wedge S} f(u,\mathcal{Y}_u,Z_u^{S,0}\cdot\mathbf{1}_{\tau\leq S_n} + Z_u^{S,L}\cdot\mathbf{1}_{\tau>S_n})du\\ &- \int_{S_n}^{T\wedge S}\left[Z_u^{S,0}\cdot\mathbf{1}_{\tau\leq S_n} + Z_u^{S,L}\cdot\mathbf{1}_{\tau>S_n}\right]dW_u\\ &+ \int_{t\wedge S}^{S_n} f(u,\mathcal{Y}_u,Z_u^{S_n,\xi_2,L})du - \int_{t\wedge S}^{S_n} Z_u^{S_n,\xi_2,L}dW_u.\end{aligned}$$

Hence, we have verified that $(\mathcal{Y}, \mathcal{Z})$ solves the BSDE (3.9). The statement of the lemma follows from the uniqueness of such a solution (Theorem 1). $\square$

We now give

Proof of Theorem 6. Let $Y^{S,\xi_2 \wedge L}$ be the solution of (3.9) with bounded terminal condition $Y_S = \xi_2 \wedge L = L \cdot \mathbf{1}_{\{\tau > S\}}$. The inequality $\xi_2 \wedge L \leq L$ implies

$$Y_t^{S,\xi_2 \wedge L} \leq Y_t^{S,L}, t \leq S.$$

This and $Y_t^{S,L} \leq n$ for $t \leq S_n$ imply that, if we define

$$Y_t^{S,\xi_2} = \lim_{L \nearrow \infty} Y_t^{S,\xi_2 \wedge L},$$

then Y^{S,ξ_2} is a classical solution of (1.1) on the time interval $[\![0, S_n]\!]$. The fact of (5.1) holds over the event $\{\xi_2 = \infty\} = \{\tau > S\}$ follows from that of Y^{S,ξ_2} being constructed by approximation from below (see [34]). For completeness, we reproduced this argument: note

$$\liminf_{t \to \infty} Y_{t \wedge S}^{S,\xi_2} \geq \liminf_{t \to \infty} Y_{t \wedge S}^{S,\xi_2 \wedge L} = \xi_2 \wedge L$$

for all L. Letting $L \nearrow \infty$ implies

$$\liminf_{t \to \infty} Y_{t \wedge S}^{S,\xi_2} \geq \xi_2.$$

In particular,

$$\lim_{t \to \infty} Y_{t \wedge S}^{S,\xi_2} = \liminf_{t \to \infty} Y_{t \wedge S}^{S,\xi_2} = \xi_2 = \infty$$

over the event $\{\tau > S\}$. This proves (5.1) over the event $\{\tau > S\}$.

It remains to prove (5.1) over the event $\{\tau \leq S\}$. Recall the process $Y^{S,\xi_2,L,n}$ of (5.2) that is the solution of (3.9) over the interval $[\![0, S]\!]$ with terminal condition $Y_S = L \cdot \mathbf{1}_{\{\tau > S_n\}}$. Given $S_n \leq S$ implies

$$L \cdot \mathbf{1}_{\{\tau > S_n\}} \geq L \cdot \mathbf{1}_{\{\tau > S\}}.$$

This and the comparison principle lead to

$$Y_t^{S,\xi_2,L} \leq Y_t^{S,\xi_2,L,n}, \quad \text{for } t \leq S.$$

Lemma 6 implies

$$Y_t^{S,\xi_2,L,n} = Y_t^{S,0}, \text{for } t \in]\!]S_n, S]\!]$$

over the event $\{\tau \leq S_n\}$. Combining the last two displays we get

$$Y_t^{S,\xi_2,L} \leq Y_t^{S,0}, \text{for } t \in]\!]S_n, S]\!]$$

over the event $\{\tau \leq S_n\}$. The right side of the last inequality doesn't depend on L. Taking limits on the left gives

$$Y_t^{S,\xi_2} \leq Y_t^{S,0}, \text{for } t \in]\!]S_n, S]\!]$$

over the event $\{\tau \leq S_n\}$. The right side of the above inequality is a classical solution of the BSDE (3.9) with 0 terminal condition. Therefore, taking limits of both sides above give

$$\limsup_{t \to \infty} Y_{t \wedge S}^{S,\xi_2} \leq \lim_{t \to \infty} Y_{t \wedge S}^{S,0} = 0.$$

By its construction, $Y^{S,\xi_2} \geq 0$. This and the last display imply

$$\lim_{t \to \infty} Y_{t \wedge S}^{S,\xi_2} = 0$$

over the event $\{\tau \leq S_n\}$. Finally, $S_n \nearrow S$ and $\mathbb{P}(\tau = S) = 0$ imply $\bigcup_{n=1}^{\infty} \{\tau \leq S_n\} = \{\tau \leq S\}$. This and the last display imply

$$\lim_{t \to \infty} Y_{t \wedge S}^{S,\xi_2} = 0 = \xi_2$$

over the event $\{\tau \leq S\}$. This completes the proof of the theorem. $\square$

CHAPTER 6

AN APPLICATION TO OPTIMAL LIQUIDATION

A special case of the BSDE studied in chapter 4 arise in the optimal liquidation problem. In this chapter, we focus on the Brownian filtration case, i.e., we assume $\mathbb{F}$ is to be the filtration generated by a single dimensional Brownian motion W.

We build on the Almgren-Chriss framework as presented in [16, Chapter 3]. Consider an investor who wants to close a position $q_0 > 0$ on a financial asset over a time interval $[0, T]$. Let q_t denote the position of the investor at time t; q is assumed to be differentiable:

$$q_t = q_0 + \int_0^t v_s ds.$$

The midprice of the asset at time t is assumed to be

$$S_t = S_0 + \bar{S}_t + kq_t, \bar{S}_t = \sigma W_t,$$

where k is the permanent market impact factor. Note that the price S_t consists of two components: $\bar{S}$ and q. The first of these represent the changes in price that is independent of the investor. The market volume is assumed to be a constant $V > 0$. The actual trading price is,

$$S_t + \frac{V}{v_t} L(v_t/V)$$

where the second term models the execution cost of the trade at time t. As in [16] we assume L to be quadratic, $L(\rho) = \eta\rho^2$ which leads to the following trading price:

$$S_t + \eta\frac{v_t}{V}.$$

The cash position generated by the trading curve q is then

$$X_t = -\int_0^t (S_s + \eta\frac{v_s}{V})v_s ds = -\int_0^t S_s v_s d_s - \int_0^t \frac{\eta}{V} v_s^2 ds. \tag{6.1}$$

As in [16, Chapter 3], integration by parts gives

$$\int_0^t S_s v_s ds = S_t q_t - S_0 q_0 - \int_0^t q_s \sigma dW_s - \frac{k}{2}(q_t^2 - q_0^2).$$

We can then write (6.1) as

$$X_t = S_0 q_0 - S_t q_t + \int_0^t q_s \sigma dW_s + \frac{k}{2}(q_t^2 - q_0^2) - \int_0^t \frac{\eta}{V} v_s^2 ds. \tag{6.2}$$

All of the terms in this expression have natural interpretations: the first two represent money made from the main price process, the third term is the gain or lost from the fluctuations in price, the fourth term is the mundane lost from the permanent impact of the trade on price and the last term is the money paid to transaction costs.

The goal of the investor is to close her position q_0 before a given terminal time $\tau > 0$ and in doing this maximizing her expected utility. This liquidation algorithm is known as "implementation shortfall" (IS). The simplest utility function that can be used in this formulation is the identity function which leads to the problem of maximizing the expectation of the cash position. *Assuming that* $\mathbb{E}[\tau] < \infty$, the only term that can be controlled in $\mathbb{E}[X_\tau]$ by choosing q are the expected transaction costs; thus, we are lead to the following control problem:

$$\min_{q \in \mathcal{A}} \mathbb{E}\left[\int_0^\tau \frac{\eta}{V}\left(\frac{dq}{dt}(s)\right)^2 ds\right], \tag{6.3}$$

where $\mathcal{A} = \{q : q(0) = q_0, q \text{ is differentiable}, q(\tau) = 0\}$. In the classical framework treated in [16, Chapter 3] the terminal time is taken to be deterministic: $\tau = T$.

For $a < 0 < b$, define $\tau_{a,b} = \inf\{t : \bar{S}_t \notin (a, b)\}$. Considering the above problem with terminal time $\tau_{a,b}$ corresponds to specifying the following constraint to the control problem: liquidate before the initial price deviates b or $-a$ from the initial price. Note that $\mathbb{E}[\tau_{a,b}] < \infty$ and therefore, indeed we have

$$\mathbb{E}[X_{\tau_{a,b}}] = S_0 q_0 - \mathbb{E}\left[\int_0^\tau \frac{\eta}{V}\left(\frac{dq}{dt}(s)\right)^2 ds\right],$$

and (6.3) is equivalent to maximizing $\mathbb{E}[X_{\tau_{a,b}}]$. The functions and BSDE computed in Section 4.3 give an explicit solution to this problem.

In this chapter we would like to consider another possibility: suppose that the investor specifies an additional target price $S_0 + u$, $u > 0$ that she considers to be attractive and would like the position to be closed completely when this price is attained. The upperbound can be specified either for $\bar{S}$ or S. To relate the problem to the BSDE in the previous chapters we focus on the case when the upperbound is specified on $\bar{S}$, the random component of the midprice that the investor cannot control. Define

$$\tau_u = \inf\{t : \bar{S}_t \geq u\}.$$

We therefore propose the constraint $q_{\tau_u} = 0$, i.e., the investor would like to close the position as soon as $\bar{S}$ hits u.

Before we propose a solution, we would like to point out an important change that happens in the nature of the problem when the terminal time is set to τ_u: because $\mathbb{E}[\tau_u] = \infty$, we no longer have

$$\mathbb{E}\left[\int_0^\tau q_s \sigma dW_s\right] = 0;$$

this implies

$$\mathbb{E}[X_{\tau_u}] \neq S_0 q_0 - \mathbb{E}\left[\int_0^\tau \frac{\eta}{V}\left(\frac{dq}{dt}(s)\right)^2 ds\right].$$

Therefore, minimization of expected transaction costs and maximization of expected payoff are no longer equivalent problems. In the rest of this section we will focus on the first problem, i.e., the minimization of transaction costs and offer a solution to this problem.

Assume $V(q_0, s) = q_0^2 U(s)$. The BSDE corresponding to the above problem reduces to

$$\frac{1}{2}\sigma^2 U_{ss} - \frac{V}{\eta}U^2 = 0. \tag{6.4}$$

We next solve the last ODE over the interval $(-\infty, u]$ with boundary condition

$$U(u) = \infty, U(-\infty) = 0. \tag{6.5}$$

Suppose $V_0 : [0, \infty) \mapsto \mathbb{R}$ satisfies (6.4) on the interval $[0, \infty)$ with boundary condition $V_0(0) = \infty, V_0(\infty) = 0$.

We note the following:

Lemma 7.

$$U(s) = V_0(-s + u)$$

satisfies (6.4) *on the interval* $(-\infty, u]$ *and the boundary condition* (6.5).

Proof.

$$\begin{aligned} U_{ss} &= \frac{d^2}{ds^2}V_0(-s+u) = -\frac{d}{ds}(V_0)_s(-s+u) \\ &= (V_0)_{ss}(-s+u) = \frac{2V}{\sigma^2\eta}V_0^2(-s+u) = \frac{2V}{\sigma^2\eta}U^2(s) \end{aligned}$$

satisfying (6.4). $U(s)$ is a reflection of V_0 in the y-axis shifted u units to the right. Therefore the domain for which $U(s)$ coincides with that of $V_0(s)$ is on minus the domain of V_0 shifted u units to the right, i.e., $-[0,\infty)+u=(-\infty,u]$.

$$U(u)=V_0(-u+u)=V_0(0)=\infty \quad \text{and} \quad U(-\infty)=V_0(\infty+u)=V_0(\infty)=0$$

satisfying (6.5). □

The next proposition gives an explicit formula for V_0:

Proposition 2. *The minimal supersolution of* (6.4) *on* $[0,\infty)$ *with boundary condition* $V_0(0)=\infty$*,* $V_0(\infty)=0$ *is*

$$V_0(s)=\frac{3\sigma^2\eta}{V}s^{-2}.$$

Proof. Let $V_0=Cs^{\alpha}$ satisfy (6.4). Then we have,

$$\frac{1}{2}\sigma^2\alpha(\alpha-1)Cs^{\alpha-2}-\frac{V}{\eta}C^2s^{2\alpha}=0.$$
$$C=\frac{\eta\sigma^2\alpha(\alpha-1)}{2V}s^{-(2+\alpha)}$$

therefore

$$V_0=\frac{\eta\sigma^2\alpha(\alpha-1)}{2V}s^{-2}.$$

Resubstituting into (6.4), we have $\alpha(\alpha-1)=6$, giving us $V_0(s)=\frac{3\sigma^2\eta}{V}s^{-2}$, satisfying (6.4) with boundary conditions $V_0(0)=\infty$, $V_0(\infty)=0$. Note that $V_n(s)=V_0(s+a_n)$ is the classical solution of (6.4) with boundary conditions $V_n(0)=n$, $V_n(\infty)=0$. Clearly, $V_n\nearrow V_0$, which proves the minimal supersolution property of V_0. □

By the above lemma and proposition, the minimal supersolution of (6.4) on $(-\infty, u]$ with boundary conditions (6.5) is

$$U(s) = \frac{3\sigma^2 V}{\eta}(u - s)^{-2}.$$

Then the value function of (6.3) for $\tau = \tau_u$ equals

$$v(s, q) = \left(\frac{q}{u - s}\right)^2 \frac{3\sigma^2 V}{\eta}.$$

By [22, Theorem 4], the corresponding optimal liquidation process is

$$q_t^* = q_0 e^{-\int_0^t U(\bar{S}_t)ds}$$

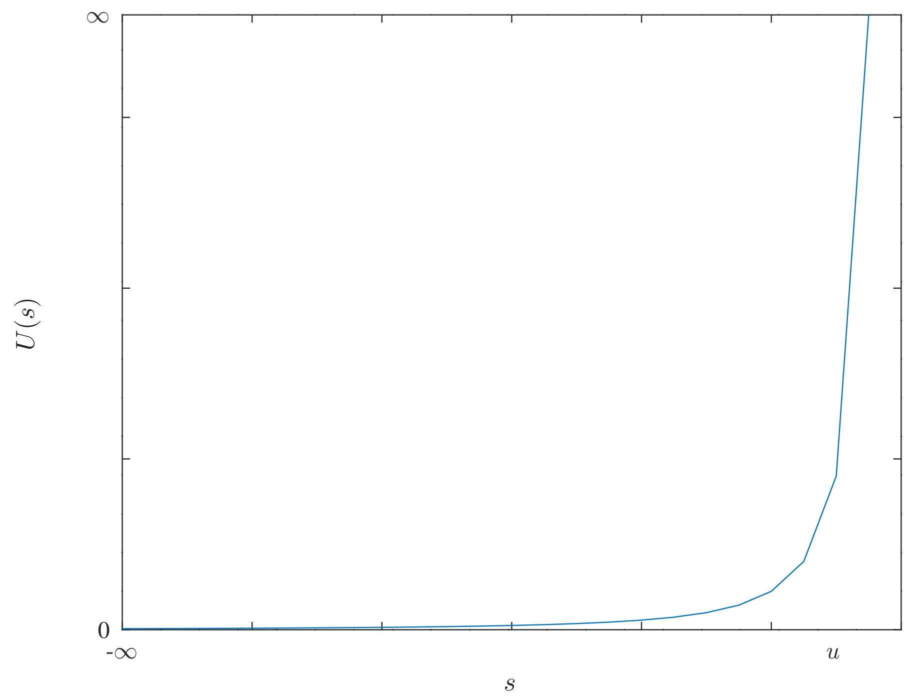

Figure 6.1: Graph of the function $U(s)$

In figure 6.1, we see that $U(s)$ is an increasing convex function which satisfies the boundary conditions $U(\infty) = 0$ and $U(u) = \infty$ as given in (6.5).

In figure 6.2, it is noted that as our $S(t)$ process approaches u, we have our open position terminating, somewhat abruptly, at τ_u, the first time $S(t)$ hits the upper boundary

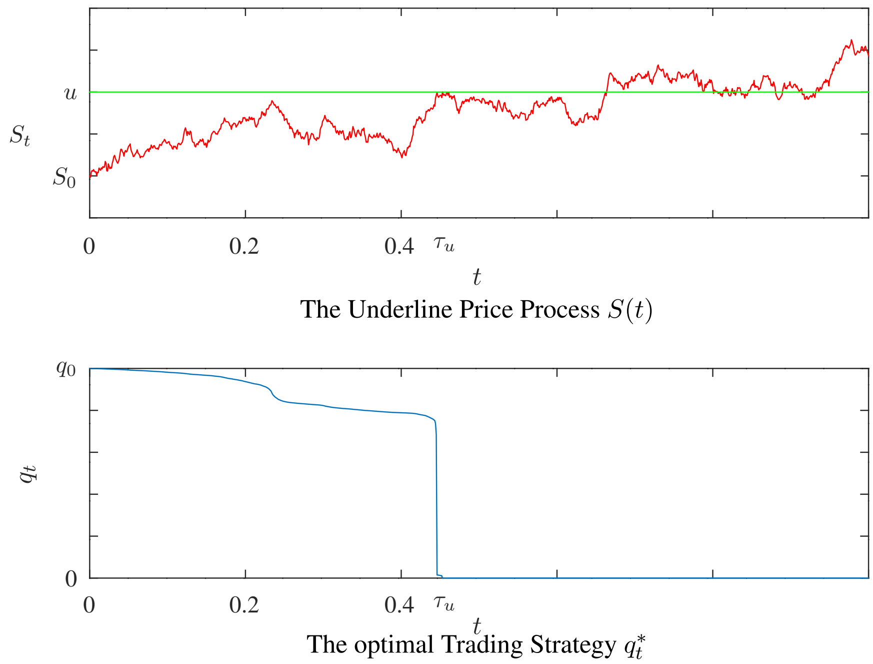

Figure 6.2: Trading Strategy qt with the $S(t)$ process

u. Conversely, in figure 6.3, $S(t)$ does not go beyond u, resulting in an incomplete dissolution of the portfolio since our target-price was not attained.

Furthermore, in figure 6.4, we consider how the strategy performed with varying volatility $(\sigma_1 = 0.01, \sigma_1 = 0.04, \sigma_1 = 0.08)$. It can be observed that the trading strategy closes the position faster in the case of a lager σ value than those with lower ones. Events of a slight hint of $S(t)$ approaching u, with a high σ value, our trading curve dissolves the portfolio almost instantly. Whereas with a low σ value, it does not budge until $S(t)$ gets significantly closer to u. Hence, with an open positions with high risk, the portfolio is closed quickly and somewhat gradually, while, though later than those with high σ, less risky ones seems to be traded of later when $S(t)$ in fact approaches u, similar to case in figure 6.2 with an abrupt trade off to the end.

Additionally, considering the impact of total trading volumes, as is depicted in figure

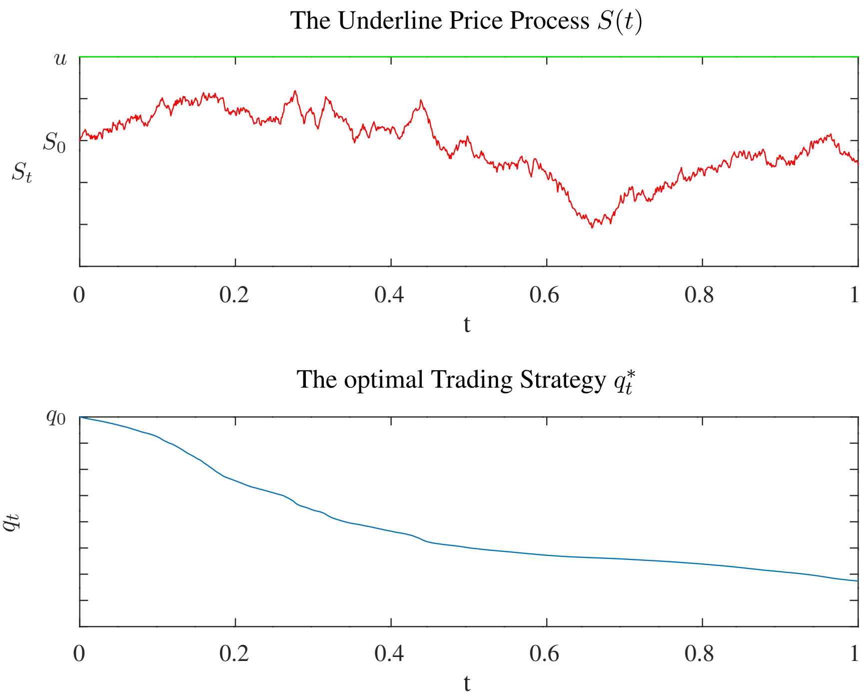

Figure 6.3: Trading Strategy qt does not liquidate completely as the process S remains below u

6.5 ($V_1 = 100, V_2 = 500, V_3 = 1000$), we see with high trading volumes, our portfolio is dissolved quickly and our trading curves are convex in nature. On-the-other-hand, with lower volumes, our trades are delayed to the end of the trading period with the curve taking a concave shape close the end.

Moreover, in figure 6.6, we see that with varying initial positions that the trading curve is a scaled identical curve of each other.

Finally, we consider the scenario with varying trading cost ($\eta_1 = 1, \eta_2 = 3, \eta_3 = 10$). Looking at figure 6.7, we observe that when there are low trading cost, our trading curve begins the liquidation process earlier hence, there is less trading volumes close to τ_u. In contrast, with higher ηs, trading is deferred with high liquefaction taking place around τ_u.

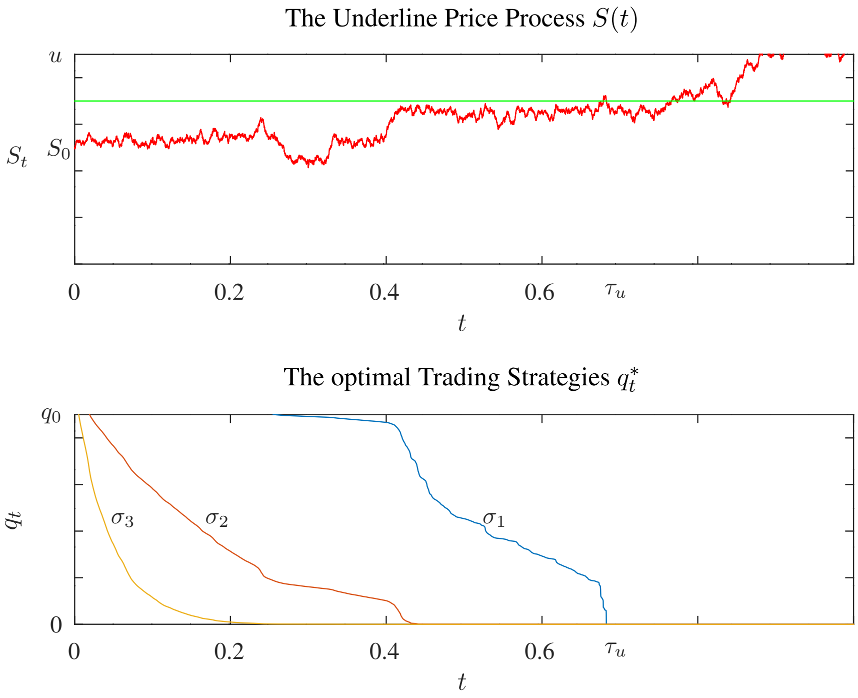

Figure 6.4: Trading Strategy with varying Volatility $(\sigma_1 < \sigma_2 < \sigma_3)$.

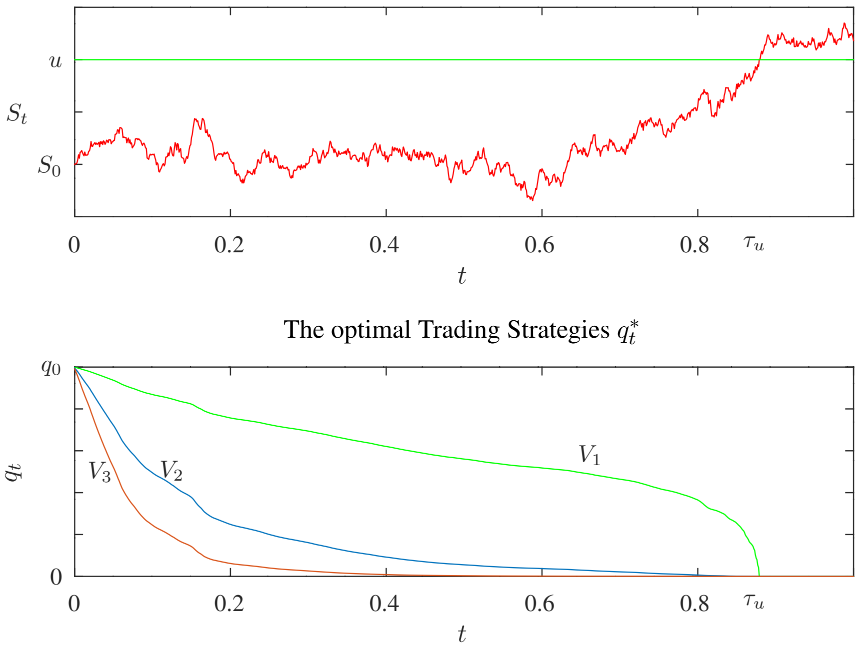

Figure 6.5: Trading Strategy with Varying Volumes $(V_1 < V_2 < V_3)$.

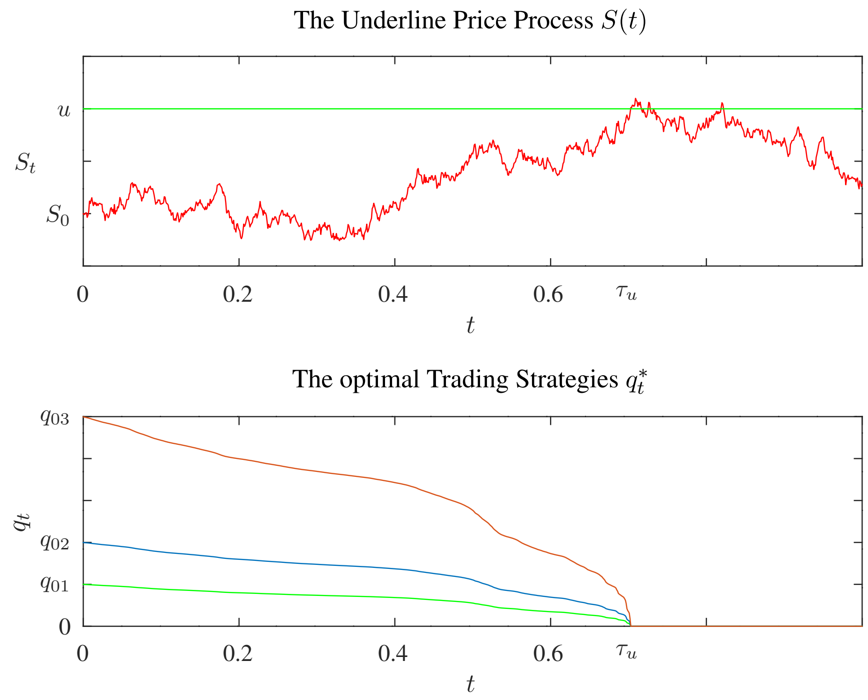

Figure 6.6: Trading Strategy with Varying Opening Position $q0$

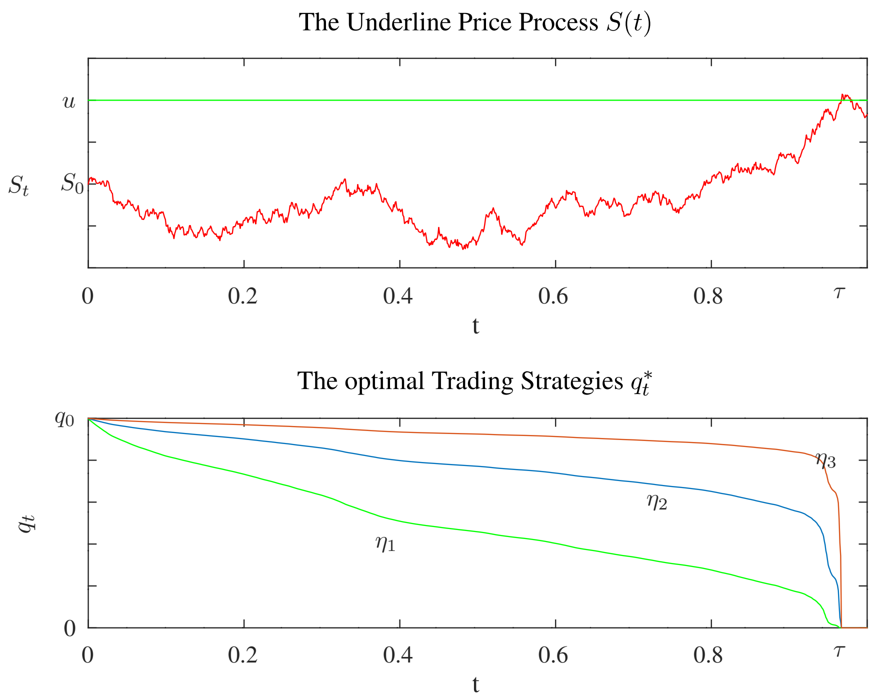

Figure 6.7: Trading Strategy with Varying Trading Cost $(\eta_1 < \eta_2 < \eta_3)$.

CHAPTER 7

CONCLUSION

The present work develops solutions to the BSDE (1.1) with random terminal time S for a range of singular terminal values. We do this by proving that the minimal supersolution is continuous at S and attains the terminal value by constructing upper-bound processes that are known to have the desired behavior at terminal time. A key ingredient of our framework and our arguments is the concept of a solvable stopping time with respect to the given BSDE and the filtration, introduced in the present work. Solvability means that the the BSDE has a supersolution with value ∞ at the given stopping time. We note that a stopping time that has a positive density around 0 is not solvable. We also note that deterministic times as well as exit times of continuous diffusion processes from smooth domains are solvable. A natural direction for future work is to further understand the concept of solvability and identify other classes of solvable/non-solvable stopping times.

In this thesis we, focused on non-Markovian terminal conditions. For furhter results on BSDE with singular terminal conditions with a solvable terminal time, we refer to [36] where results on Markovian terminal times as well as their connection to solution of PDE with singular boundary conditions are also presented. The same work also presents results on the continuity of Y at terminal time S based on the left continuity of the filtration at time S.

In Chapters 4 and 5, we focus on terminal values of the form $\xi_1 = \infty \cdot \mathbf{1}_{\{\tau<S\}}$ and $\xi_2 = \infty \cdot \mathbf{1}_{\{\tau>S\}}$; the arguments in the present work can be generalized to terminal conditions of the form $\xi_1 = \infty \cdot \mathbf{1}_{\{\tau<S\}} + A \cdot \mathbf{1}_{\{\tau\geq S\}}$ and $\xi_2 = \infty \cdot \mathbf{1}_{\{\tau>S\}} + A \cdot \mathbf{1}_{\{\tau\leq S\}}$, where A is a sufficiently integrable random variable. Such an extension is given in [2] for deterministic terminal times. Another formulation explored in [2] for a deterministic terminal time T is terminal conditions of the form $\xi = \infty \cdot \mathbf{1}_{A_T}$ where A_t is a decreasing sequence of events adapted to the filtration $\mathcal{F}_t$ and is continuous in probability at time T. A study of this formulation for solvable terminal times remains for future work.

Technical analysis in finance consists of basing trading strategies on the use of resistance and support levels as discussed in [4, 9, 12, 28]; perhaps the thresholds used in the liquidation algorithm in Chapter 6 can be selected using these levels. An idea that may be pursued is to allow u to vary in time and chosen to be trend lines or indicators. Another idea for future research is the generalization of the computations in 6 to more general price and trading dynamics.

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